

CONIFOLD TRANSITIONS VIA AFFINE GEOMETRY AND MIRROR SYMMETRY

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ABSTRACT. Mirror symmetry of Calabi-Yau manifolds can be understood via a Legendre duality between a pair of certain affine manifolds with singularities called tropical manifolds. In this article, we study conifold transitions from the point of view of Gross and Siebert [10, 11, 12]. We introduce the notions of tropical nodal singularity, tropical conifolds, tropical resolutions and smoothings. We interpret known global obstructions to the complex smoothing and symplectic small resolution of compact nodal Calabi-Yaus in terms of certain tropical 2-cycles containing the nodes in their associated tropical conifolds. We prove that the existence of such cycles implies the simultaneous vanishing of the obstruction to smoothing the original Calabi-Yau *and* to resolving its mirror. We conjecture that the existence of these cycles also implies that the tropical conifold can be resolved and its dual tropical conifold can be smoothed, thus showing that the mirror of the resolution is a smoothing. We prove the conjecture for certain configurations of nodes and study some interesting examples.

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1. INTRODUCTION

A geometric transition between a pair of smooth varieties is the process of deforming the first variety to a singular one and then obtaining the second one by resolving the singularities. The first variety is called a smoothing and the second one a resolution. In [24], Morrison conjectures that, in certain circumstances, mirror symmetry should map a pair of smooth Calabi-Yau manifolds, related by a geometric transition, to another pair, also related by a geometric

transition but with the roles reversed, so that the mirror of a smoothing should be a resolution and viceversa. This idea is supported by evidences and examples. Morrison also suggests that a new understanding of this phenomenon could come from the SYZ interpretation of mirror symmetry as a duality of special Lagrangian torus fibrations. Building on ideas of Hitchin, Kontsevich-Soibelman and others on the SYZ conjecture [17], [19], [20], Gross and Siebert show that mirror pairs of Calabi-Yau manifolds can be constructed from a Legendre dual pair of affine manifolds with singularities and polyhedral decompositions, also called tropical manifolds [10, 11, 12]. In this article we consider the special case of conifold transitions. We introduce the notion of tropical conifold, i.e. of tropical manifold with nodes, and we show that the smoothing/resolution process also has a natural description in this context. Indeed the smoothing of a tropical conifold simultaneously induces a resolution of its mirror, but in general the process is obstructed. To study global obstructions we introduce the notion of tropical cycle. In terms of the original Calabi-Yau varieties, we prove that the existence of tropical cycles containing the nodes implies the simultaneous vanishing of the obstructions to the smoothing of one variety and to the resolution of its mirror. We conjecture that the existence of such cycles containing the nodes should also allow the resolution/smoothing of the tropical conifold. This would show that the smoothing and the resolution are themselves mirror pairs in the sense of Gross-Siebert. We prove the conjecture for some special configurations of nodes and we apply the results to some interesting examples. In terms of the Lagrangian fibrations constructed by the authors [4], these results also determine criteria to establish if and how a Lagrangian fibration “survives” a conifold transition.

1.1. Conifold transitions. A node is the 3-fold singularity with local equation $xy - zw = 0$. A small resolution of a node has a \mathbb{P}^1 as its exceptional cycle, with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. The smoothing of a node (i.e. $xy - zw = \epsilon$) produces a Lagrangian 3-sphere as a vanishing cycle. A conifold transition is the geometric transition associated with a 3-fold with nodal singularities (i.e. a “conifold”). It was proved by Friedman and Tian [5], [30] that a compact complex conifold can be smoothed to a complex manifold if and only if the exceptional cycles of a small resolution satisfy a “good relation” in homology (see equation (16)). Similarly, on the symplectic side, it was shown by Smith, Thomas and Yau [28] that a “symplectic conifold” has a symplectic (small) resolution, with symplectic exceptional cycles, if and only if the vanishing cycles of a smoothing satisfy a good relation. These two results are a manifestation of the idea that the mirror of a complex smoothing should be a symplectic resolution. Indeed this idea has been used to find the mirror partner of one of the Calabi-Yau’s involved in a conifold transition from the knowledge of the mirror of the other Calabi-Yau (see [24], [2]).

1.2. SYZ conjecture. Mirror symmetry is usually computed when the Calabi-Yau manifold is the generic fibre of a family $\psi : \chi \rightarrow \mathbb{C}$, where the special fibre $\chi_0 = \psi^{-1}(0)$ is highly degenerate. More precisely one requires that χ_0

has maximally unipotent monodromy (or χ_0 is what the physicists call a large complex structure limit point). The Strominger-Yau-Zaslow (SYZ) conjecture [29] claims that mirror Calabi-Yau pairs X and \check{X} should admit “dual” special Lagrangian fibrations, $f : X \rightarrow B$ and $\check{f} : \check{X} \rightarrow B$. This original idea has been revised by Gross-Wilson [6], [13] and Kontsevich-Soibelman [19], who claimed that special Lagrangian fibrations should exist only in some limiting sense as the fibre $\chi_s = \psi^{-1}(s)$ approaches the singular fibre χ_0 (see also the survey paper [8]). This “limiting fibration” can be described in terms of a certain structure on the base B of the fibration. Here B is a real manifold and the structure on B should contain information concerning the complex and symplectic structure of the Calabi-Yau manifold. Moreover, this data contains intrinsically a duality given by a Legendre transform. The important fact is that the structure on B should allow the “reconstruction” of the original Calabi-Yau. This is known as the reconstruction problem. Therefore, finding the mirror of a given family $\psi : \chi \rightarrow \mathbb{C}$ of Calabi-Yau’s becomes the process of constructing B , with its structure, applying the Legendre transform to obtain the dual base \check{B} , with dual structure, and then reconstructing the mirror family via some reconstruction theorem. For instance, in dimension 2, Kontsevich and Soibelman [20], construct a rigid analytic $K3$ from an affine structure on S^2 with 24 punctures.

1.3. Tropical manifolds and mirror symmetry. In all dimensions, this program has been completed by Gross and Siebert in a sequence of papers [10], [11], [12]. On B they consider the structure of an integral affine manifold with singularities and polyhedral decompositions. Roughly this means that B is obtained by gluing a set of n -dimensional integral convex polytopes in \mathbb{R}^n by identifying faces via integral affine transformations (this is the polyhedral decomposition, denoted \mathcal{P}). Then, at the vertices v of \mathcal{P} one defines a fan structure, which identifies the tangent wedges of the polytopes meeting at v with the cones of a fan Σ_v in \mathbb{R}^n . There are some additional compatibility conditions required by the fans defined at vertices belonging to a common face of \mathcal{P} . For a certain codimension 2 closed subset $\Delta \subset B$, this structure determines an atlas on $B_0 = B - \Delta$ such that the transition maps are integral affine transformations (this is the integral affine structure). The set Δ , called the discriminant locus, is the set of singularities of the affine structure. An additional crucial piece of data is a polarization, which consists of a so-called “strictly convex multivalued piecewise linear function” ϕ on B . Such a ϕ is specified by the data of a strictly convex piecewise linear function ϕ_v defined on every fan Σ_v , plus compatibility conditions between ϕ_v and ϕ_w for vertices v and w belonging to a common face. All these data, which we denote by the triple (B, \mathcal{P}, ϕ) , are also called a polarized tropical manifold. The “discrete Legendre transform” associates to (B, \mathcal{P}, ϕ) another triple $(\check{B}, \check{\mathcal{P}}, \check{\phi})$. Essentially, at a vertex v of B , the fan Σ_v and function ϕ_v provide an n -dimensional polytope \tilde{v} , by the standard construction in toric geometry. Two polytopes \tilde{v} and \tilde{w} , associated to vertices v and w on a common edge of \mathcal{P} , can be glued together

along a face using the compatibilities between the pairs (Σ_v, ϕ_v) and (Σ_w, ϕ_w) . This gives \check{B} and the polyhedral decomposition $\check{\mathcal{P}}$. The fan structure and function $\check{\phi}$ at the vertices of $\check{\mathcal{P}}$ come from the n -dimensional polytopes of \mathcal{P} essentially using the inverse construction.

In order to have satisfactory reconstruction theorems it is necessary to put further technical restrictions on (B, \mathcal{P}, ϕ) . Gross and Siebert define such conditions and call them “positivity and simplicity”. For convenience, we will say that a polarized tropical manifold is smooth if it satisfies these nice conditions. In particular, in the 3-dimensional case smoothness of B amounts to the fact that Δ is a 3-valent graph and the vertices can be of two types: “positive” or “negative”, depending on the local monodromy of the affine structure. The Gross-Siebert reconstruction theorem [12], ensures that given a smooth polarized tropical manifold (B, \mathcal{P}, ϕ) , it is possible to construct a toric degeneration $\psi : \chi \rightarrow \mathbb{C}$ of Calabi-Yau varieties, such that B is the dual intersection complex of the singular fibre χ_0 . This means that χ_0 is the union of toric varieties glued along toric divisors. The toric varieties are precisely those given by the fans Σ_v . The compatibility between the fans allow the toric varieties to be glued together to form χ_0 . The difficult part in the Gross-Siebert theorem consists in the reconstruction of the whole family $\psi : \chi \rightarrow \mathbb{C}$. The mirror family $\check{\psi} : \check{\chi} \rightarrow \mathbb{C}$ is obtained by applying the reconstruction theorem to the Legendre dual $(\check{B}, \check{\mathcal{P}}, \check{\phi})$. Gross and Siebert give various evidences that this construction coincides with mirror symmetry.

The integral affine structure on $B_0 = B - \Delta$ implies the existence of a local system $\Lambda^* \subset T^*B_0$, whose fibres $\Lambda_b \cong \mathbb{Z}^n$ are maximal lattices in $T_b^*B_0$. Then one can form the n -torus bundle $X_{B_0} = T^*B_0/\Lambda^*$ over B_0 . The standard symplectic form on T^*B_0 descends to X_{B_0} and the projection $f_0 : X_{B_0} \rightarrow B_0$ is a Lagrangian torus fibration. In [4], we proved that if B is a 3-dimensional smooth tropical manifold then one can form a symplectic compactification of X_{B_0} . This is a symplectic manifold X_B , containing X_{B_0} as a dense open subset, together with a Lagrangian fibration $f : X_B \rightarrow B$ which extends f_0 . This is done by inserting suitable singular Lagrangian fibres over points of Δ . Topologically the compactification X_B is based on the one found by Gross in [7]. It is expected that X_B should be diffeomorphic to a smooth fibre χ_s of the family $\psi : \chi \rightarrow \mathbb{C}$ in the Gross-Siebert reconstruction theorem, whose dual intersection complex is $(\check{B}, \check{\mathcal{P}}, \check{\phi})$. This result has been announced in [9], Theorem 0.1. A complete proof for the quintic 3-fold in \mathbb{P}^4 is found in [7]. We also expect that X_B should be symplectomorphic to χ_s with a suitable Kähler form, although there is no proof of this yet.

1.4. Summary of the results. In dimension 3, smoothness of B ensures that the general fibre χ_s of $\psi : \chi \rightarrow \mathbb{C}$ is smooth. In this article we introduce the notion of (polarized) tropical conifold, in which the discriminant locus is allowed to have 4-valent vertices. Such vertices, which we call (tropical) nodes, are of two types: negative and positive. Away from these nodes, a tropical conifold

is a smooth tropical manifold. We believe that the Gross-Siebert reconstruction theorem can be extended also to tropical conifolds, but the general fibre χ_s should be a variety with nodes. This is hinted by the fact that the local conifold $xy - wz = 0$ has a pair of torus fibrations which induce on the base B the same structure as in a neighborhood of positive or negative nodes. In fact, we show that if B is a tropical conifold, then X_{B_0} can be topologically compactified to a topological conifold X_B (i.e. a singular topological manifold with nodal singularities). An interesting observation is that the Legendre transform of a positive node is the negative node. In particular we also have the mirror conifold $X_{\tilde{B}}$. This extends topological mirror symmetry of [7] to conifolds. Then we give a local description of the smoothing and resolution of a node in a tropical conifold (see Figures 9 and 10). It turns out that the Legendre dual of a resolution is indeed a smoothing. At the topological level this was already observed by Gross [6] and Ruan [26], who also discusses a global example. The interesting question is global: given a compact tropical conifold, can we simultaneously resolve or smooth its nodes? We give a precise procedure to do this. It turns out that the smoothing of nodes in a tropical conifold simultaneously induces the resolution of the nodes in the mirror. What are the obstructions to the tropical resolution/smoothing? For this purpose we define the notion of tropical 2-cycle inside a tropical conifold. These objects resemble the usual notion of a tropical surface as defined for instance by Mikhalkin in [22]. A tropical 2-cycle is given by a space S and an embedding $j : S \rightarrow B$ with some additional structure. The space S has various types of interior and boundary points. For instance at the generic points, S is locally Euclidean, at the codimension 1 points S is modeled on the tropical line times an interval and at codimension 2 points S is modeled on the tropical plane (see Figure 13) and so on. In Theorem 7.3 we prove that if $j(S)$ contains tropical nodes, then both the vanishing cycles associated to the nodes in X_B and the exceptional curves associated to the nodes in $X_{\tilde{B}}$ satisfy a good relation. Thus obstructions vanish on both sides of mirror symmetry. The results of Friedman, Tian and Smith-Thomas-Yau then lead us to Conjecture 8.2, which states that a set of nodes in the tropical conifold B can be resolved if and only if it is contained in the union of tropical 2-cycles. We give a proof of this conjecture for certain special configurations of nodes in Theorems 8.3, 8.5, 8.7 and in Corollary 8.4. We expect that, if unobstructed, then the resolution/smoothing of a set of nodes should give a tropical manifold whose associated Gross-Siebert family is the resolution/smoothing of the nodes in the family associated with the tropical conifold. This is true for the topological reconstruction.

Finally we apply these results to specific examples. We consider the case of Schoen's Calabi-Yau [27], which is a fibred product of two rational elliptic surfaces. A corresponding tropical manifold has been described by Gross in [9]. It is possible to modify the example in many ways so that we obtain a tropical conifold with various nodes. We show how these nodes can be resolved/smoothed and thus obtain new tropical manifolds. The interesting fact

is that this procedure automatically produces the mirror families via discrete Legendre transform and the reconstruction theorems.

1.5. Notations. We denote the convex hull of a set of points q_1, \dots, q_r in \mathbb{R}^n by $\text{Conv}(q_1, \dots, q_r)$. Given a set of vectors $v_1, \dots, v_r \in \mathbb{R}^n$ the cone spanned by these vectors is the set

$$\text{Cone}(v_1, \dots, v_r) = \left\{ \sum_{j=1}^r t_j v_j \mid t_j \geq 0, j = 1, \dots, r \right\}.$$

2. AFFINE MANIFOLDS WITH POLYHEDRAL DECOMPOSITIONS

We give an informal introduction to affine manifolds with singularities and polyhedral decompositions. We refer to [10] for precise definitions and proofs.

2.1. Affine manifolds with singularities. Let $M \cong \mathbb{Z}^n$ be a lattice and define $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and let

$$\text{Aff}(M) = M \rtimes \text{Gl}(\mathbb{Z}, n)$$

be the group of integral affine transformations of $M_{\mathbb{R}}$. If M and M' are two lattices, then $\text{Aff}(M, M')$ is the \mathbb{Z} -module of integral affine maps between $M_{\mathbb{R}}$ and $M'_{\mathbb{R}}$. Recall that an integral affine structure \mathcal{A} on an n -manifold B is given by an open cover $\{U_i\}$ and an atlas of charts $\phi_i : U_i \rightarrow M_{\mathbb{R}}$ whose transition maps $\phi_j \circ \phi_i^{-1}$ are in $\text{Aff}(M)$. An integral affine manifold is a manifold B with an integral affine structure \mathcal{A} . A continuous map $f : B \rightarrow B'$ between two integral affine manifolds is integral affine if, locally, f is given by elements of $\text{Aff}(M, M')$.

An *affine manifold with singularities* is a triple, (B, Δ, \mathcal{A}) , where the B is an n -manifold, $\Delta \subseteq B$ is a closed subset such that $B_0 = B - \Delta$ is dense in B and \mathcal{A} is an integral affine structure on B_0 . The set Δ is called the *discriminant locus*. A continuous map $f : B \rightarrow B'$ of integral affine manifolds with singularities is integral affine if $f^{-1}(B'_0) \cap B_0$ is dense in B and $f|_{f^{-1}(B'_0) \cap B_0} : f^{-1}(B'_0) \cap B_0 \rightarrow B'_0$ is integral affine. Furthermore f is an isomorphism of integral affine manifolds with singularities if $f : (B, \Delta) \rightarrow (B', \Delta')$ is a homeomorphism of pairs.

2.2. Parallel transport and monodromy. Given an affine manifold B , let $(U, \phi) \in \mathcal{A}$ be an affine chart with coordinates u_1, \dots, u_n . Then the tangent bundle TB (resp. cotangent bundle T^*B) has a flat connection ∇ defined by

$$\nabla \partial_{u_j} = 0 \quad (\text{resp. } \nabla du_j = 0),$$

for all $j = 1, \dots, n$ and all charts $(U, \phi) \in \mathcal{A}$. Then parallel transport along loops based at $b \in B$ gives the monodromy representation $\tilde{\rho} : \pi_1(B) \rightarrow \text{Gl}(T_b B)$. Gross-Siebert also introduce the notion of holonomy representation, which is denoted ρ and has values in $\text{Aff}(T_b B)$, $\tilde{\rho}$ coincides with the linear part of ρ . In the case of an affine manifold with singularities (B, Δ, \mathcal{A}) , the monodromy representation is $\tilde{\rho} : \pi_1(B_0) \rightarrow \text{Gl}(T_b B_0)$.

Integrality implies the existence of a maximal integral lattice $\Lambda \subset TB_0$ (resp. $\Lambda^* \subset T^*B_0$) defined by

$$(1) \quad \Lambda|_U = \text{span}_{\mathbb{Z}}\langle \partial_{u_1}, \dots, \partial_{u_n} \rangle \quad (\text{resp. } \Lambda^*|_U = \text{span}_{\mathbb{Z}}\langle du_1, \dots, du_n \rangle).$$

We can therefore assume that $\tilde{\rho}$ has values in $\text{Gl}(\mathbb{Z}, n)$.

2.3. Polyhedral decompositions. Rather than recalling here the precise definition of an *integral affine manifold with singularities and polyhedral decompositions* (i.e. Definition 1.22, op. cit.), it is better to recall the standard procedure to construct them (see Construction 1.26, op. cit.). We start with a finite collection \mathcal{P}' of n -dimensional integral convex polytopes in $M_{\mathbb{R}}$. The manifold B is formed by gluing together the polytopes of \mathcal{P}' via integral affine identifications of their proper faces. Then B has a cell decomposition whose cells are the images of faces of the polytopes of \mathcal{P}' . Denote by \mathcal{P} this set of cells. For clarity of exposition we make the simplifying assumption that given a cell $\tau \in \mathcal{P}$, then every polytope $\sigma' \in \mathcal{P}'$ has at most one face $\tau' \subset \sigma'$ whose image is τ , i.e. we assume there are no identifications between faces belonging to the same polytope. Moreover we assume that B is a compact manifold without boundary. We now construct the integral affine atlas \mathcal{A} on B . First of all, the interior of each maximal cell of \mathcal{P} can be regarded as the domain of an integral affine chart, since it comes from the interior of a polytope in $M_{\mathbb{R}}$. To define a full atlas we need charts around points belonging to lower dimensional cells. In fact this will be possible only after removing from B a set Δ' which we now define. Let $\text{Bar}(\mathcal{P})$ be the first barycentric subdivision of \mathcal{P} . Then define Δ' to be the union of all simplices of $\text{Bar}(\mathcal{P})$ not containing a vertex of \mathcal{P} (i.e. 0-dimensional cells) or the barycenter of a maximal cell. For a vertex $v \in \mathcal{P}$, let W_v be the union of the interiors of all simplices of $\text{Bar}(\mathcal{P})$ containing v . Then W_v is an open neighborhood of v and

$$\{W_v \mid v \text{ is a vertex of } \mathcal{P}\} \cup \{\text{Int}(\sigma) \mid \sigma \in \mathcal{P}_{\max}\}$$

forms a covering of $B - \Delta'$. A chart on the open set W_v is given by a *fan structure* at the vertex v , which we now define. Consider a pair (v', σ') , where $\sigma' \in \mathcal{P}'$ and v' is a vertex of σ' which is mapped to v . Let $T(v', \sigma')$ be the tangent wedge of σ' at v' , which is a convex rational polyhedral cone in $M_{\mathbb{R}}$. Suppose (v', σ') and (v'', σ'') are two pairs, with v' and v'' both mapped to v and such that σ' and σ'' are identified along faces $\tau' \subset \sigma'$ and $\tau'' \subset \sigma''$ containing v' and v'' respectively. Then the tangent wedges $T(v', \sigma')$ and $T(v'', \sigma'')$ can be identified in a unique way along the tangent wedges $T(v', \tau')$ and $T(v'', \tau'')$. The space formed by gluing all the tangent wedges of pairs (v', σ') , with v' mapped to v , via these identifications is homeomorphic to \mathbb{R}^n and can be regarded as the tangent space $T_v B$. Moreover we have an inclusion $W_v \subset T_v B$, since σ' is always contained in $T(v', \sigma')$. Now choose a complete rational polyhedral fan Σ_v in $M_{\mathbb{R}}$, in the usual sense of toric geometry, such that there is a one to one correspondence between the tangent wedges $T(v', \sigma')$ and the maximal cones of Σ_v , which we denote $T(v', \sigma') \mapsto \sigma_{v'}$. Furthermore there should exist integral affine isomorphisms i_{σ} between $T(v', \sigma')$ and $\sigma_{v'}$ which are compatible with

the above identifications between the faces of the cones. This gives a uniquely defined homeomorphism $T_v B \rightarrow M_{\mathbb{R}}$ and hence a chart on W_v . This is the fan structure at v . It is clear that this construction gives an integral affine atlas on $B - \Delta'$. In many cases the set Δ' is too crude and the affine structure can be extended to a larger set than $B - \Delta'$. This can be done as follows. Notice that Δ' is a union of codimension 2 simplices. If τ is one such simplex of Δ' , let U_τ be a small neighborhood of the interior $\text{Int}(\tau)$ of τ such that $\pi_1(U_\tau - \text{Int}(\tau)) \cong \mathbb{Z}$. Then let Δ be the union of simplices $\tau \subset \Delta'$ such that the local monodromy $\tilde{\rho} : \pi_1(U_\tau - \text{Int}(\tau)) \rightarrow \text{Gl}(\mathbb{Z}, n)$ is non trivial. In Proposition 1.27 of op. cit. it is proved that the affine structure on $B - \Delta'$ constructed above can be extended to $B - \Delta$.

The polyhedral decomposition \mathcal{P} is said to be *toric* if for any cell $\sigma \in \mathcal{P}$, there is a neighborhood U_σ of $\text{Int}(\sigma)$ and an integral affine submersion $S_\sigma : U_\sigma \rightarrow M'_{\mathbb{R}}$ where M' is a lattice of rank equal to the codimension of σ and $S_\sigma(U_\sigma \cap \sigma) = 0$. This condition is crucial in the Gross-Siebert program since it is used to construct the central fibre of a toric degeneration. Moreover it is necessary also in the definition of the discrete Legendre transform.

2.4. Local properties of monodromy. The monodromy representation of affine manifolds with polyhedral decompositions has some useful distinguished properties, which we now describe. First of all notice that Δ does not intersect the interior of maximal cells, i.e. it is contained in the codimension 1 skeleton. Let τ be a cell of \mathcal{P} of codimension at least 1, then, it is shown in op. cit. Proposition 1.29 that the tangent space to τ is monodromy invariant with respect to the local monodromy near τ . More precisely, there exists a neighborhood U_τ of $\text{Int}(\tau)$ such that, if $b \in \tau - \Delta$ and $\gamma \in \pi_1(U_\tau - \Delta, b)$, then $\tilde{\rho}(\gamma)(w) = w$ for every w tangent to τ in b . Moreover (see op. cit. Proposition 1.32) the polyhedral subdivision is toric if and only if for every τ there exists a neighborhood U_τ of $\text{Int}(\tau)$ such that, if $b \in \tau - \Delta$ and $\gamma \in \pi_1(U_\tau - \Delta, b)$, then $\tilde{\rho}(\gamma)(w) - w$ is tangent to τ for every $w \in T_b B_0$.

2.5. Quotient fans. If the polyhedral decomposition is toric (see above), then to every cell $\tau \in \mathcal{P}$ one can associate a complete fan Σ_τ in $M'_{\mathbb{R}}$ as follows. Let $b \in \text{Int}(\tau) - \Delta$, then to every σ such that $\tau \subset \sigma$, one can associate the tangent wedge of σ at b . This can be viewed as a convex rational polyhedral cone inside $T_b B_0$ (with lattice structure given by Λ). The union of all such cones forms a complete fan in $T_b B_0$, which in fact is the pull back of a complete fan in the quotient $T_b B_0 / T_b \tau$. Let Σ_τ be such a fan. The toric condition ensures that the quotient spaces $T_b B_0 / T_b \tau$ are independent of $b \in \tau - \Delta$, in fact they can be all identified with $M'_{\mathbb{R}}$ via parallel transport along paths contained in a suitably small neighborhood U_τ of $\text{Int}(\tau)$. The local properties of monodromy, assuming the toric condition, imply that this identification is independent of the chosen path. Hence also Σ_τ is independent of b and can be assumed to be a fan in $M'_{\mathbb{R}}$.

2.6. Examples. In the following examples B will be allowed to have boundary or even to be constructed using unbounded polytopes. The construction above

can be easily adapted to these cases (e.g. the fans at boundary points will not be complete).

Example 2.1. (The focus-focus singularity)

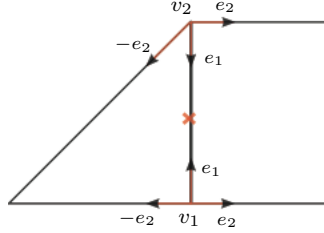


FIGURE 1.

Here the dimension is $n = 2$. The set \mathcal{P}' is given by two polytopes: a standard simplex and a square $[0, 1] \times [0, 1]$. Glue them along one edge to form B (see Figure 1). Let e be the common edge, along which we have glued the polytopes, and let v_1 and v_2 be the vertices of e . The discriminant locus Δ consists of one point, the barycenter of e . Now we define the fan structure at v_1 and v_2 . Consider the fan in \mathbb{R}^2 whose 2-dimensional cones are two adjacent quadrants, i.e. $\text{Cone}(e_1, e_2)$ and $\text{Cone}(e_1, -e_2)$, where $\{e_1, e_2\}$ is the standard basis of \mathbb{R}^2 . Then the fan structure at v_j , $j = 1, 2$, identifies the tangent wedges of the the two polytopes with these two cones, in such a way that the primitive tangent vector to e at v_j is mapped to e_1 (see Figure 1). Consider a loop γ which starts at v_1 , goes into the square, passes through v_2 and comes back to v_1 while passing inside the triangle. One can easily calculate that $\tilde{\rho}(\gamma)$, computed with respect to the basis $\{e_1, e_2\}$, as depicted in Figure 1, is the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

The singular point Δ in this example is called the focus-focus singularity.

Example 2.2. (Generic singularity) This a 3-dimensional example and it is just the product of the previous example by $[0, 1]$. Here Δ consists of a segment.

Example 2.3. (The negative vertex)

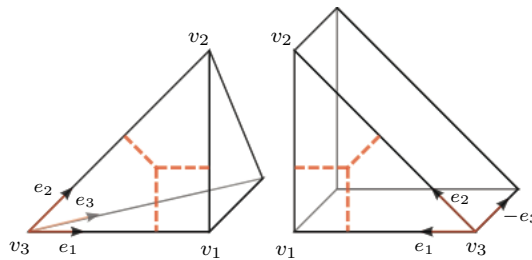


FIGURE 2.

Here $n = 3$. Let P^n be the standard simplex in \mathbb{R}^n . The set \mathcal{P}' consists of two polytopes: P^3 and $P^2 \times P^1$ which we glue by identifying the triangular face $P^2 \times \{0\}$ with a face of P^3 . In Figure 2 we have labeled the vertices of these two faces by v_1, v_2, v_3 and the identification is done by matching the vertices with the same labeling. Now consider the fan in \mathbb{R}^3 whose cones are two adjacent octants (i.e. $\text{Cone}(e_1, e_2, e_3)$ and $\text{Cone}(e_1, e_2, -e_3)$, where $\{e_1, e_2, e_3\}$ is the standard basis of \mathbb{R}^3). At every vertex v_j identify the tangent wedges of the two polytopes with these two cones, in such a way that the tangent wedge to the common face is mapped to $\text{Cone}(e_1, e_2)$. The discriminant locus Δ is the “Y” shaped figure depicted in (red) dashed lines in Figure 2. Now let γ_j be the path going from v_3 to v_j by passing into P^3 and then coming back to v_3 by passing into $P^2 \times P^1$. It can be easily shown that $\tilde{\rho}(\gamma_1)$ and $\tilde{\rho}(\gamma_2)$ are given respectively by the matrices

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

The vertex of Δ in this example is called the *negative vertex*.

Example 2.4. (The positive vertex) In this case $B = \mathbb{R}^2 \times [0, 1]$ with poly-

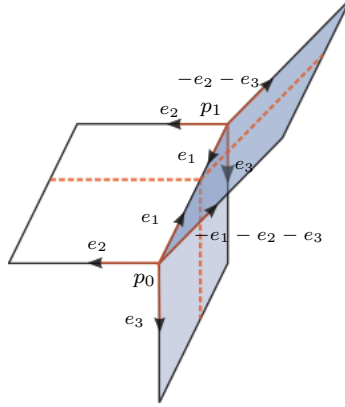


FIGURE 3.

hedral decomposition given by the following unbounded polytopes (see Figure 3):

$$Q_1 = \{x \geq \max\{y, 0\}, z \in [0, 1]\}, \quad Q_2 = \{y \geq \max\{x, 0\}, z \in [0, 1]\},$$

$$Q_3 = \{x \leq 0, y \leq 0, z \in [0, 1]\}.$$

Let Σ_0 be the fan whose maximal cones are

$$\text{Cone}(e_1, e_3, -e_1 - e_2 - e_3), \quad \text{Cone}(e_1, e_2, -e_1 - e_2 - e_3), \quad \text{Cone}(e_1, e_2, e_3).$$

Then the fan structure at p_0 identifies the tangent wedges of Q_1, Q_2 and Q_3 at p_0 with the first, second and third cone respectively. Now let Σ_1 be the fan

whose maximal cones are

$$\text{Cone}(e_1, e_3, -e_2 - e_3), \quad \text{Cone}(e_1, e_2, -e_2 - e_3), \quad \text{Cone}(e_1, e_2, e_3).$$

The fan structure at p_1 identifies the tangent wedges of Q_1, Q_2, Q_3 at p_1 with the first, second and third cone respectively. Now let γ_j , $j = 1, 2$, be the loop which starts at p_0 , goes to p_1 by passing inside Q_3 and then comes back to p_0 by passing inside Q_j . Then we have that $\tilde{\rho}(\gamma_1)$ and $\tilde{\rho}(\gamma_2)$ are given respectively by the following matrices

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The vertex of Δ in this example is called the *positive vertex*.

2.7. MPL-functions. A multi-valued piecewise linear (MPL) function on an affine manifold with singularities B and polyhedral subdivision \mathcal{P} generalizes the notion of piecewise linear function on a fan Σ in toric geometry. In fact MPL-functions on B , restricted to a neighborhood of every vertex $v \in \mathcal{P}$ become piecewise linear functions on the fan Σ_v via the fan structure. Let $U \subset B$ be an open subset. A continuous function $f : U \rightarrow \mathbb{R}$ is said to be (*integral*) *affine* if it is (integral) affine when restricted to $U \cap B_0$. The sheaf of integral affine functions (or just affine) is denoted by $\mathcal{A}ff(B, \mathbb{Z})$ (resp. $\mathcal{A}ff_{\mathbb{R}}(B, \mathbb{R})$). Notice, for example, that an affine function defined in a neighborhood of the singularity in Example 2.1 must be constant along the edge e which contains it. An (integral) *piecewise linear* (PL) function on U is a continuous function $f : U \rightarrow \mathbb{R}$ which is (integral) affine when restricted to $U \cap \text{Int}(\sigma)$ for every maximal cell $\sigma \in \mathcal{P}$ and satisfies the following property: for any $y \in U$, $y \in \text{Int}(\sigma)$ for some $\sigma \in \mathcal{P}$, there exists a neighborhood V of y and an (integral) affine function f on V such that $\phi - f$ is zero on $V \cap \text{Int}(\sigma)$. Notice that this latter property implies that a PL-function on a neighborhood of the singularity in Example 2.1 is constant when restricted to the edge e . The sheaf of integral (or just affine) PL-functions is denoted by $\mathcal{PL}_{\mathcal{P}}(B, \mathbb{Z})$ (resp. $\mathcal{PL}_{\mathcal{P}, \mathbb{R}}(B, \mathbb{R})$).

Observe that when \mathcal{P} is a toric polyhedral subdivision, then a PL-function on B satisfies the following property. Given $\sigma \in \mathcal{P}$ and a point $y \in \text{Int}(\sigma)$, then, in a small neighborhood U of y , there is an affine function f such that $\phi - f$ is zero on $U \cap \text{Int}(\sigma)$. This implies that $\phi - f$ descends to a PL-function (in the sense of toric geometry) on the quotient fan Σ_{σ} defined in §2.5. We denote this function by ϕ_{σ} and we call it the *quotient function*.

The sheaf $\mathcal{MP}\mathcal{L}_{\mathcal{P}}$ of integral *MPL-functions* is defined by the following exact sequence of sheaves

$$0 \longrightarrow \mathcal{A}ff(B, \mathbb{Z}) \longrightarrow \mathcal{PL}_{\mathcal{P}}(B, \mathbb{Z}) \longrightarrow \mathcal{MP}\mathcal{L}_{\mathcal{P}} \longrightarrow 0.$$

Given a toric polyhedral subdivision \mathcal{P} on B , an MPL-function ϕ on B is said to be (*strictly*) *convex* with respect to \mathcal{P} if ϕ_{τ} is a (strictly) convex piecewise linear function on the fan Σ_{τ} for every $\tau \in \mathcal{P}$. We can now give the following

Definition 2.5. A *(polarized) tropical manifold* is a triple (B, \mathcal{P}, ϕ) where B is an integral affine manifold with singularities, \mathcal{P} a toric polyhedral decomposition and ϕ a strictly convex MPL-function with respect to \mathcal{P} (the polarization).

It is worth to point out that this notion of tropical manifold differs from other notions appearing elsewhere in the literature, such as Mikhalkin's tropical varieties in [22]. Gross-Siebert's tropical manifolds can be seen as ambient spaces where Mikhalkin's tropical varieties can be embedded.

2.8. The discrete Legendre transform. Given a tropical manifold (B, \mathcal{P}, ϕ) , the discrete Legendre transform produces a second tropical manifold $(\check{B}, \check{\mathcal{P}}, \check{\phi})$. Topologically the pair $(\check{B}, \check{\Delta})$ is the same as (B, Δ) and the decomposition $\check{\mathcal{P}}$ is the standard dual cell decomposition. What changes is the affine structure.

Given a vertex $v \in \mathcal{P}$, the fan structure at v and the function ϕ give a fan Σ_v in $M_{\mathbb{R}}$ and a strictly convex piecewise linear function ϕ_v . These data give the standard n -dimensional Newton polytope:

$$\check{v} = \{x \in M_{\mathbb{R}}^* \mid \langle x, y \rangle \geq -\phi_v(y) \ \forall y \in M_{\mathbb{R}}\},$$

where $M^* = \text{Hom}(M, \mathbb{Z})$. Recall that there is an inclusion reversing correspondence between k -dimensional cones of Σ_v and $n - k$ -dimensional faces of \check{v} . We have our first piece of data

$$\check{\mathcal{P}}' = \{\check{v} \mid v \text{ a vertex of } \mathcal{P}\}.$$

We now have to explain how to glue together all these polytopes to form \check{B} . Suppose that τ is an edge of \mathcal{P} having v as one vertex and w as the other one. We have the corresponding fan Σ_{τ} , the strictly convex piecewise linear function ϕ_{τ} and its $(n - 1)$ -dimensional Newton polytope $\check{\tau}$. On the other hand, the tangent wedge of τ at v is a 1-dimensional cone of Σ_v and it thus corresponds to an $(n - 1)$ -dimensional face of \check{v} , denote it by \check{v}_{τ} . Similarly, we have that the tangent wedge of τ at w corresponds to an $(n - 1)$ -dimensional face of \check{w} , denote it by \check{w}_{τ} . It is not hard to show that the properties of the function ϕ are such that \check{v}_{τ} and \check{w}_{τ} are both integral affine isomorphic to $\check{\tau}$, via a uniquely determined isomorphism. Thus \check{v} and \check{w} can be glued along \check{v}_{τ} and \check{w}_{τ} . It can be shown that the space produced from $\check{\mathcal{P}}'$ via these gluings is a manifold \check{B} homeomorphic to B , with induced cell decomposition $\check{\mathcal{P}}$.

It remains to define the affine structure by assigning a fan structure at all vertices of $\check{\mathcal{P}}$. Notice that a vertex of $\check{\mathcal{P}}$ is the dual of a maximal polytope $\sigma \in \mathcal{P}$, thus we denote it by $\check{\sigma}$. Notice also that the maximal cells of $\check{\mathcal{P}}$ containing $\check{\sigma}$ are in one to one correspondence with the vertices of σ . Now given a vertex v of a polytope σ in $M_{\mathbb{R}}$, recall the definition of normal cone of σ at v :

$$C_{v,\sigma} = \{x \in M_{\mathbb{R}}^* \mid \langle x, y - v \rangle \geq 0, \ \forall y \in \sigma\}.$$

It is a standard fact of toric geometry that the normal cones to the vertices of σ fit into a fan $\check{\Sigma}_{\sigma}$ called the normal fan of σ . This is the fan giving the fan structure at $\check{\sigma}$. In fact it can be easily verified the tangent cone at $\check{\sigma}$ to the

maximal cell \check{v} is isomorphic to $C_{v,\sigma}$. Notice that this construction also gives a strictly convex PL-function on $\check{\Sigma}_\sigma$:

$$\check{\phi}_\sigma(y) = -\inf\{\langle x, y \rangle, x \in \sigma\}$$

for $y \in M_\mathbb{R}^*$. Then all these functions defined for all vertices $\check{\sigma}$ of $\check{\mathcal{P}}$ patch together to give a well defined polarization $\check{\phi}$ on $(\check{B}, \check{\mathcal{P}})$.

As an example, it can be easily shown that the negative vertex (Example 2.3) and the positive one (Example 2.4) are related to each other via a discrete Legendre transform with respect to suitably chosen polarizations.

2.9. The Gross-Siebert reconstruction theorem. Gross and Siebert consider tropical manifolds which satisfy a further set of technical conditions which they call “positive and simple”. To avoid confusion with other uses of the word “positive” in this paper, we will say that a (polarized) tropical manifold is *smooth* if it is “positive and simple” in the Gross-Siebert sense. In dimension $n = 2$ or 3 , smoothness amounts to the following. If $n = 2$, Δ consists of a finite set of points and every point of Δ has a neighborhood which is integral affine isomorphic to a neighborhood of the focus-focus singularity in Example 2.1. If $n = 3$, then Δ is a trivalent graph such that: (a) every point in the interior of an edge of Δ has a neighborhood which is integral affine isomorphic to a neighborhood of a singular point in Example 2.2 (b) every vertex of Δ has neighborhood which is integral affine isomorphic to a neighborhood of the “positive vertex” in Example 2.4 or of the “negative vertex” in Example 2.3. In this definition we should also allow Δ to be curved. In fact in Examples 2.2, 2.4 and 2.3 we could take Δ to be made of curved lines lying inside the same 2-dimensional face and we would still have a well defined affine structure on $B - \Delta$. For the rest of this paper we restrict to dimensions $n = 2$ or 3 .

In §4 of [10], Gross and Siebert consider a toric degeneration $\psi : \chi \rightarrow \mathbb{C}$ of varieties, with a relatively ample line bundle \mathcal{L} on χ and they associate to it its dual intersection complex which has the structure of a tropical manifold (B, \mathcal{P}, ϕ) . A toric degeneration has the property that the central fibre $\chi_0 = \psi^{-1}(0)$ is obtained from a disjoint union of toric varieties by identifying pairs of irreducible toric Weil divisors. This information is encoded in (B, \mathcal{P}, ϕ) , in fact to every vertex $v \in \mathcal{P}$ we associate the toric variety S_v given by the fan Σ_v . Now to every edge τ , connecting vertices v and w , the intersection between S_v and S_w is the toric divisor D_τ given by the fan Σ_τ . The polarization ϕ determines $\mathcal{L}|_{\chi_0}$. In [12] they prove the important reconstruction theorem

Theorem 2.6. (Gross-Siebert) Every compact and smooth polarized tropical manifold arises as the dual intersection complex of a toric degeneration.

In dimension $n = 3$ the generic fibre of the toric degeneration constructed in the above theorem is a smooth manifold and if B is a 3-sphere then it is also Calabi-Yau. If we consider the discrete Legendre transform $(\check{B}, \check{\mathcal{P}}, \check{\phi})$, then we can reconstruct the toric degeneration $\check{\psi} : \check{\chi} \rightarrow \mathbb{C}$. Gross and Siebert claim that this family is mirror symmetric to the family $\psi : \chi \rightarrow \mathbb{C}$ and provide many

evidences of this. For instance Gross, in [9], shows that Batyrev-Borisov mirror pairs [1] of Calabi-Yau manifolds arise in this way (see also the articles [14], [15] and [16]).

3. LAGRANGIAN FIBRATIONS

Now consider an integral affine manifold with singularities B with discriminant locus Δ and recall the definition (1) of the lattice $\Lambda^* \subset T^*B_0$. Then we can define the $2n$ -dimensional manifold

$$X_{B_0} = T^*B_0/\Lambda^*,$$

which, together with the projection $f_0 : X_{B_0} \rightarrow B_0$, forms a T^n fibre bundle. The standard symplectic form on T^*B_0 descends to X_{B_0} and the fibres of f_0 are Lagrangian. Clearly the monodromy representation $\tilde{\rho}$ associated to the flat connection on T^*B_0 is also the monodromy of the local system Λ^* . A “symplectic compactification” of X_{B_0} is a symplectic manifold X_B , together with a Lagrangian fibration $f : X_B \rightarrow B$ such that we have the following commutative diagram

$$(2) \quad \begin{array}{ccc} X_{B_0} & \hookrightarrow & X_B \\ \downarrow & & \downarrow \\ B_0 & \hookrightarrow & B \end{array}$$

where the vertical arrows are the fibrations and the upper rightarrow is a symplectomorphism. We will restrict our attention to the 3-dimensional case $n = 3$. In this case, in [4] we proved that X_B and f can be constructed under the assumption that B is smooth (in the sense of §2.9). In fact the precise statement of the result in [4] is slightly more delicate due to the fact that near negative vertices the discriminant locus Δ has to be perturbed so that it has a small codimension 1 part. We will explain more about this later. Our symplectic construction of X_B is based on the topological construction carried out by Gross in [7]. It is expected that X_B should be symplectomorphic to a generic fibre of the reconstructed toric degeneration of Theorem 2.6 with some Kähler form, where the mirror \check{B} is the dual intersection complex. In [7], in the case of the quintic 3-fold in \mathbb{P}^4 , Gross proved that X_B is diffeomorphic to a generic quintic and $X_{\check{B}}$ is diffeomorphic to its mirror (see also [9], Theorem 0.1).

The fibration f will have three types of singular fibres: *generic-singular fibres* over edges of Δ ; *positive fibres* and *negative fibres* respectively over positive and negative vertices of Δ . Let $U \subseteq B$ be a small open neighborhood, homeomorphic to a 3-ball, of either an edge, a positive or negative vertex. The idea is to find standard local models of fibrations $f_U : X_U \rightarrow U$, such that if we let $U_0 = U \cap B_0$ and $X_{U_0} = f_U^{-1}(U_0)$, then $f_U : X_{U_0} \rightarrow U_0$ has the structure of a T^3 -fibre bundle, i.e. $X_{U_0} = \mathcal{E}_{U_0}/\Lambda_{U_0}$, where \mathcal{E}_{U_0} is a rank 3-vector bundle over U_0 and Λ_{U_0} is a maximal lattice. Then, topologically, in order to glue $f_U : X_U \rightarrow U$ to $f_0 : X_{B_0} \rightarrow B_0$ it is enough to show that $\mathcal{E}_{U_0}/\Lambda_{U_0}$ and $f_0^{-1}(U_0)$ are isomorphic as T^3 -bundles, i.e. that they have the same monodromy. When f_U is Lagrangian then an isomorphism is provided by action angle coordinates.

3.1. Local models. We sketch the construction of the local models, for details see [7]§2 and [4]. We will denote local fibrations by $f : X \rightarrow U$, instead of the more cumbersome $f_U : X_U \rightarrow U$. The examples have the following structure. Let Y be a manifold of dimension 5 and let $\Sigma \subseteq Y$ be an oriented 2-dimensional submanifold and $Y' = Y - \Sigma$. Suppose there is a 6 dimensional manifold X with an S^1 action such that $X/S^1 \cong Y$. If we denote by $\pi : X \rightarrow Y$ the projection, we assume that Σ is the image of the fixed point set of the action. Moreover, if $X' = \pi^{-1}(Y')$, then we assume that $\pi : X' \rightarrow Y'$ is a principal S^1 -bundle over Y' such that the Chern class c_1 , evaluated on a small unit sphere in the fibre of the normal bundle of Σ is ± 1 . We will only consider the case $Y = U \times T^2$, with U homeomorphic to a 3-ball. Then the trivial T^2 fibration $P : Y \rightarrow U$ can be lifted to a T^3 fibration $f := P \circ \pi : X \rightarrow U$ with discriminant locus $\Delta := P(\Sigma)$. One can readily see that for $b \in \Delta$, the singularities of the fibre X_b occur along $\Sigma \cap P^{-1}(b)$.

Remark 3.1. The local structure of X and its S^1 action is the following. Any $p \in \Sigma$ has a neighborhood $V \subset Y$ such that $V \cong \mathbb{R} \times \mathbb{C}^2$, $\pi^{-1}(V) \cong \mathbb{C}^3$ such that in these coordinates $\pi(z_1, z_2, z_3) = (|z_1|^2 - |z_2|^2, z_1 z_2, z_3)$. The S^1 -action is

$$(3) \quad \xi \cdot (z_1, z_2, z_3) = (\xi z_1, \xi^{-1} z_2, z_3)$$

for all $\xi \in S^1$. Notice that $V \cap \Sigma$ is identified with the set $\{(0, 0, z) \in \mathbb{R} \times \mathbb{C}^2\}$.

Example 3.2 (Generic singular fibration). This example is the model for the fibration over a neighborhood U of an edge of Δ . Let $U = D \times (0, 1)$, where $D \subset \mathbb{C}$ is the unit disc, and let $Y = T^2 \times U$. Let $\Sigma \subset Y$ be a cylinder defined as follows. Let e_2, e_3 be a basis of $H_1(T^2, \mathbb{Z})$. Let $S^1 \subset T^2$ be a circle representing the homology class e_3 . Define $\Sigma = S^1 \times \{0\} \times (0, 1)$. Now one can construct $\pi : X \rightarrow Y$ as above and we define $f = P \circ \pi : X \rightarrow B$ where $P : Y \rightarrow B$ is the projection. Then f is a T^3 fibration with singular fibres homeomorphic to S^1 times a fibre of type I_1 , i.e. a pinched torus, lying over $\Delta := \{0\} \times (0, 1)$. If e_1 is an orbit of the S^1 action, one can take e_1, e_2, e_3 as a basis of $H_1(X_b, \mathbb{Z})$, where X_b is a regular fibre.

In this basis the monodromy associated to a simple loop around Δ is

$$(4) \quad T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Notice that e_1 and e_3 are monodromy invariant. If one also chooses a section, then the set of smooth fibres $X_0 = f^{-1}(U - \Delta)$ has the structure of a T^3 -fibre bundle \mathcal{E}/Λ .

An explicit Lagrangian fibration with this topology is defined as follows. Let

$$(5) \quad X = \{(z_1, z_2, z_3) \in \mathbb{C}^2 \times \mathbb{C} \mid z_1 z_2 - 1 \neq 0, z_3 \neq 0\}$$

with the standard symplectic form induced from \mathbb{C}^3 and $U = \mathbb{R}^3$. Define $f : X \rightarrow U$ to be

$$(6) \quad f(z) = (\log |z_1 z_2 - 1|, |z_1|^2 - |z_2|^2, \log |z_3|),$$

The discriminant locus is $\Delta = \{x_1 = x_2 = 0\}$ and the S^1 action is defined as in (3). The quotient space is identified with $\mathbb{R} \times (\mathbb{C}^*)^2$ as in Remark 3.1. Notice that f is actually invariant with respect to the T^2 action given by

$$(7) \quad \xi \cdot (z_1, z_2, z_3) = (\xi_1 z_1, \xi_1^{-1} z_2, \xi_2 z_3),$$

for $\xi = (\xi_1, \xi_2) \in T^2$. The second and third components of f give the moment map with respect to this action.

Example 3.3 (Negative fibration). This example is a model of a fibration over a neighborhood U of a negative vertex of Δ . We give two possible versions, which are topologically equivalent. The first one is defined as follows. Let

$$\bar{Y} = T^2 \times \mathbb{R}^2$$

Define $\Delta \subset \mathbb{R}^2$ to be a “Y” shaped subset and write $\Delta = \{b_0\} \cup \Delta_1 \cup \Delta_2 \cup \Delta_3$ where b_0 is the vertex of Δ and the Δ_i are the legs of Δ . For instance we could define $b_0 = (0, 0)$, $\Delta_1 = \{(-t, 0) \mid t > 0\}$, $\Delta_2 = \{(0, -t) \mid t > 0\}$ and $\Delta_3 = \{(t, t) \mid t > 0\}$. Fix a basis e_2, e_3 for $H_1(T^2, \mathbb{Z})$. Define $\Sigma \subset \bar{Y}$ to be a “pair of pants” lying over Δ such that for $i = 1, 2, 3$, $\Sigma \cap (T^2 \times \Delta_i)$ is the cylinder $S^1 \times \Delta_i$, where S^1 is a circle in T^2 representing the classes $-e_3$, $-e_2$ and $e_2 + e_3$ respectively. These legs can be glued together over the vertex b_0 of Δ in such a way that $\Sigma \cap (T^2 \times \{b_0\})$ is a figure eight curve. Now let

$$Y = \bar{Y} \times \mathbb{R} = T^2 \times \mathbb{R}^3$$

and identify \bar{Y} with $\bar{Y} \times \{0\}$ and $\Delta \subset \mathbb{R}^2$ with $\Delta \times \{0\} \subset \mathbb{R}^2 \times \mathbb{R}$. Now consider a space X and a map $\pi : X \rightarrow Y$ with the properties described above, in particular with $\Sigma \subset \bar{Y}$ as the image of the fixed point sets of the S^1 action on X . Consider the trivial T^2 fibration $P : Y \rightarrow \mathbb{R}^3$ given by projection. The composition $f = P \circ \pi$ is 3-torus fibration. For $b \in \Delta$ the fibre $f^{-1}(b)$ is singular along $P^{-1}(b) \cap \Sigma$, which is a circle when $b \in \Delta_i$, or the aforementioned figure eight when $b = b_0$. Thus the fibres over Δ_i are homeomorphic to $I_1 \times S^1$, whereas the central fibre, X_{b_0} , is singular along a nodal curve. We can take as a basis of $H_1(X_b, \mathbb{Z})$, $e_1(b), e_2(b), e_3(b)$, where e_2 and e_3 are the 1-cycles in $P^{-1}(b) = T^2$ as before and e_1 is a fibre of the S^1 -bundle. In this basis, the monodromy matrices associated to suitable loops about the legs Δ_i are

$$(8) \quad T_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We now describe the second version. It is defined over

$$X = \{z_1 z_2 + z_3 - 1 \neq 0\} \cap \{z_1 z_2 - z_3 \neq 0\}$$

by the function

$$f(z_1, z_2, z_3) = (|z_1|^2 - |z_2|^2, \log |z_1 z_2 + z_3 - 1|, \log |z_1 z_2 - z_3|).$$

If we consider the S^1 action (3) and the associated projection π onto the quotient space. Then $f = P \circ \pi$, where

$$P : (t, v) \mapsto (t, \log |u_1 + u_2 - 1|, \log |u_1 - u_2|).$$

which is T^2 fibration. Notice that the image under π of the fixed point set is the surface

$$\Sigma = \{t = 0, u_1 = 0\}.$$

The discriminant locus Δ is $P(\Sigma)$, which can be seen to be a codimension

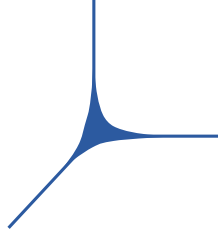


FIGURE 4.

1 thickening of the “Y” shaped figure (i.e. the amoeba of a line). It is not difficult to perturb this fibration so that the ends of the legs of Δ become pinched to codimension 2 (see Figure 4). With more effort one can make this fibration into a piece-wise smooth Lagrangian one. We emphasize that from a topological point of view this latter version of negative fibration is equivalent to the first version with Δ of codimension 2. In fact one can show that the surface Σ in this latter version is isotopic to a surface which maps to a genuine codimension 2 “Y” shaped Δ constructed as in the first version. Also in this example the set of smooth fibres $X_0 = f^{-1}(\mathbb{R}^3 - \Delta)$ has the structure of a 3-torus bundle \mathcal{E}/Λ .

We now describe the fibration over a neighborhood of a positive vertex.

Example 3.4 (Positive fibration). In this case we give an explicit fibration which turns out to be also Lagrangian. Here $X = \{z_1 z_2 z_3 - 1 \neq 0\}$, with standard symplectic form induced from \mathbb{C}^3 . The fibration $f : X \rightarrow \mathbb{R}^3$ is defined by

$$(9) \quad f(z_1, z_2, z_3) = (\log |z_1 z_2 z_3 - 1|, |z_1|^2 - |z_2|^2, |z_1|^2 - |z_3|^2).$$

Here $\Delta = \{(0, t, t), t \geq 0\} \cup \{(0, 0, -t), t \geq 0\} \cup \{(0, -t, 0), t \geq 0\}$. Notice that f is invariant with respect to the T^2 action

$$(10) \quad \xi \cdot (z_1, z_2, z_3) = (\xi_1^{-1} z_1, \xi_2^{-1} z_2, \xi_1 \xi_2 z_3).$$

The singular fibre over the vertex of Δ can be described as $T^2 \times S^1$ after one of the T^2 's is collapsed to a point. The presence of a T^2 action implies that the monodromy group generated by monodromies around the three legs of Δ leaves a 2-dimensional subspace fixed. In fact the monodromy matrices around the legs are inverse transpose of the matrices in (8).

Remark 3.5. As we explained, these local models can be glued to X_{B_0} in order to obtain a symplectic manifold X_B and a Lagrangian fibration $f : X_B \rightarrow B$ fitting into the diagram (2). The above local models can also be modified so that the discriminant locus Δ is “curved”, i.e. the edges of Δ bend inside the two dimensional monodromy invariant planes that contain them. All these local models $f_U : X_U \rightarrow U$ also have smooth Lagrangian sections $\sigma_U : U \rightarrow X_U$ and the gluing can be done so that these sections will match with the standard zero section $\sigma_0 : B_0 \rightarrow X_{B_0}$. Therefore we can assume that $f : X_B \rightarrow B$ also has a fixed smooth Lagrangian section, which we continue to denote $\sigma_0 : B \rightarrow X_B$ and regard it as the zero section.

Remark 3.6. Given a 3-dimensional tropical manifold (B, \mathcal{P}, ϕ) and its dual $(\check{B}, \check{\mathcal{P}}, \check{\phi})$ we can also define the torus bundle

$$\check{X}_{B_0} = TB_0/\Lambda.$$

If B is smooth, then \check{X}_{B_0} and $X_{\check{B}_0}$ are topologically equivalent torus bundles. This follows from the fact that the Legendre transform exchanges negative and positive vertices. Therefore the monodromy of \check{X}_{B_0} around a vertex of Δ is the same as the monodromy of $X_{\check{B}_0}$ around the same vertex. This also implies that topologically \check{X}_{B_0} can be compactified by gluing a positive fibre over a negative vertex and viceversa. We denote the compactification by \check{X}_B . Clearly \check{X}_B is homeomorphic to $X_{\check{B}}$.

4. A REVIEW OF CONIFOLD TRANSITIONS

4.1. Local geometry. Recall that an ordinary double point or *node* is an isolated 3-fold singularity with local equation in \mathbb{C}^4 given by

$$(11) \quad z_1 z_2 - z_3 z_4 = 0$$

We call the “local conifold” the 3-dimensional affine variety X_0 defined by this equation. It has a small resolution $\pi : X \rightarrow X_0$, where

$$(12) \quad X = \{(z, [t_1 : t_2]) \in \mathbb{C}^4 \times \mathbb{P}^1 \mid t_1 z_1 = t_2 z_3, t_2 z_2 = t_1 z_4\}$$

and π is the projection onto \mathbb{C}^4 . Recall that X is the total space of the bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. A smoothing of X_0 , is given by

$$(13) \quad Y_\epsilon = \{z_1 z_2 - z_3 z_4 = \epsilon\}.$$

The symplectic form on X_0 and Y_ϵ is the restriction of the standard symplectic form on \mathbb{C}^4 . The symplectic structure on X is induced by the symplectic structure on $\mathbb{C}^4 \times \mathbb{P}^1$ which, in coordinates (z, t) with $t = t_2/t_1$, is given by

$$(14) \quad \sum_{i=1}^4 dz_i \wedge d\bar{z}_i + \delta \frac{dt \wedge d\bar{t}}{(1 + |t|^2)^2}$$

where δ is the area of $\pi^{-1}(0) \cong \mathbb{P}^1$.

Observe that after the following change of coordinates

$$z_1 \mapsto w_1 + iw_2, \quad z_2 \mapsto w_1 - iw_2, \quad z_3 \mapsto -w_3 - iw_4, \quad z_4 \mapsto w_3 - iw_4$$

we can write $X_0 = \{\sum_{j=1}^4 w_j^2 = 0\}$ and $Y_\epsilon = \{\sum_{j=1}^4 w_j^2 = \epsilon\}$. Therefore the set $\{\text{Im } w = 0\}$ in Y_ϵ defines a Lagrangian 3-sphere of radius ϵ , which is the vanishing cycle of the node. The cotangent bundle of S^3 can be written as

$$T^*S^3 = \{(u, v) \in \mathbb{R}^4 \times \mathbb{R}^4 \mid |u| = 1, \langle u, v \rangle = 0\}$$

with canonical symplectic form $dv \wedge du$. The important fact is that there is a symplectomorphism

$$(15) \quad \psi : X_0 - \{0\} \rightarrow T^*S^3 - \{v = 0\}$$

given explicitly by

$$w_j = x_j + iy_j \mapsto \left(\frac{x_j}{|x|}, -|x|y_j \right).$$

More generally a complex conifold is a 3-dimensional projective (or compact Kähler) variety \bar{X} whose singular locus is a finite set of nodes p_1, \dots, p_k . Given a conifold, one can try to find a small resolution $\pi : X \rightarrow \bar{X}$, which replaces every node with an exceptional \mathbb{P}^1 , or a smoothing \tilde{X} , which replaces a node with a 3-sphere. Passing from X to \tilde{X} (or viceversa) is called a *conifold transition*. Small resolutions and smoothings can always be done topologically, but there are obstructions if one wishes to preserve either the complex or symplectic (Kähler) structure.

4.2. Complex smoothings. A complex small resolution of a set of nodes in a complex conifold always exists, in the sense that a small resolution always has a natural complex structure such that the exceptional \mathbb{P}^1 's are complex submanifolds. Notice also that there are 2^k -choices of resolutions, where k is the number of nodes, in fact in (12) one could write X exchanging z_3 and z_4 . Although locally this is just a change of variables, globally the two choices may even give topologically distinct resolutions (they are related by a flop). On the other hand finding a complex analytic smoothing of \bar{X} is obstructed and the obstructions were studied by Friedman [5] and Tian [30]. Assuming that the resolution X satisfies the $\partial\bar{\partial}$ -lemma, Friedmann and Tian proved the following. Denote by $C_j \subset X$, $j = 1, \dots, k$ the exceptional \mathbb{P}^1 's of a resolution, then a complex smoothing of \bar{X} exists if and only if for some resolution the classes $[C_j] \in H_2(X; \mathbb{Z})$ satisfy

$$(16) \quad \sum_j \lambda_j [C_j] = 0, \quad \text{with } \lambda_j \neq 0 \text{ for all } j$$

Such a relation is called a *good relation*.

4.3. Symplectic resolutions. In general, even if \bar{X} is projective, the resolution X does not have a natural Kähler form. This suggests that symplectic resolutions are obstructed. The problem was studied by Smith-Thomas-Yau in [28]. First they show that a symplectic conifold can always be symplectically smoothed (op. cit. Theorem 2.7). More precisely, they define a symplectic conifold \bar{X} roughly, to be a singular topological manifold with a finite set of singular nodal points p_1, \dots, p_k , with a symplectic structure $\bar{\omega}$ on $\bar{X} - \{p_1, \dots, p_k\}$

which is sufficiently close to the symplectic structure on $X_0 - \{0\}$ near the singularities. Then they show that the smoothing \tilde{X} has a natural symplectic form by replacing the singular points with Lagrangian 3-spheres L_1, \dots, L_k (called the *vanishing cycles*) using the symplectomorphism (15). On the other hand, on the resolution X we can pull back the symplectic form on \tilde{X} , but this is obviously degenerate along exceptional \mathbb{P}^1 's. One can attempt to add to $\pi^*\bar{\omega}$ a small closed two form η such that $\pi^*\bar{\omega} + \eta$ is modeled on (14) near the exceptional \mathbb{P}^1 's, thus turning them into symplectic submanifolds, but this has global obstructions. Smith-Thomas-Yau prove the following “mirror” of the Friedman-Tian result. In a smoothing \tilde{X} of \bar{X} , the classes $[L_j] \in H_3(\tilde{X}, \mathbb{Z})$ satisfy a good relation:

$$\sum_j \lambda_j [L_j] = 0, \quad \text{with } \lambda_j \neq 0 \text{ for all } j,$$

if and only if there is a symplectic structure on one of the 2^k choices of resolutions X such that the resulting exceptional \mathbb{P}^1 's are symplectic.

4.4. Local collars. Let us now make some observations which will be useful in the proof of Theorem 7.3. Consider the following 4-dimensional submanifold with boundary of T^*S^3 :

$$N_0 = \{(u, v) \in T^*S^3 \mid v_1 = -\lambda u_2, v_2 = \lambda u_1, v_3 = -\lambda u_4, v_4 = \lambda u_3; \lambda \geq 0\}.$$

Observe that N_0 can be regarded as half a real line bundle and we have

$$\partial N_0 = S^3.$$

Under the symplectomorphism ψ given in (15), N_0 is the image of the complex surface

$$\Sigma_0 = \{w_1 = iw_2, w_3 = iw_4\}$$

In the z coordinates we have

$$(17) \quad \Sigma_0 = \{z_2 = 0, z_4 = 0\}.$$

Clearly we could have also defined Σ_0 to be one of the following $\{z_2 = 0, z_3 = 0\}$, $\{z_1 = 0, z_4 = 0\}$ or $\{z_1 = 0, z_3 = 0\}$, then Σ_0 would still be mapped to a 4-manifold bounding S^3 , differing from N_0 only by a change in signs in the defining equations. We may think of N_0 or Σ_0 as a “local collar” near the vanishing cycle. In particular, suppose that inside a symplectic conifold \bar{X} there exists a 4-manifold S (without boundary) containing nodes p_1, \dots, p_k , such that in local coordinates z_1, \dots, z_4 around each node, S coincides with Σ_0 . Then in a smoothing \tilde{X} , S lifts to a 4-dimensional manifold with boundary whose boundary is the union of the vanishing cycles L_1, \dots, L_k (see [28]).

We can similarly define a local collar near the exceptional \mathbb{P}^1 in X . In fact consider the following set:

$$(18) \quad P_0 = \{(z, \bar{z}, r, s) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \mid rs = |z|^2\}.$$

Then clearly P_0 is contained in X_0 and it can be verified that $\pi^{-1}(P_0)$ is a real 3-dimensional submanifold with boundary of X , homeomorphic to $\mathbb{P}^1 \times [0, +\infty)$,

whose boundary is the exceptional curve. Therefore suppose that inside a complex conifold \tilde{X} we can find a subset S , containing nodes p_1, \dots, p_k , such that $S - \{p_1, \dots, p_k\}$ is a real 3-dimensional submanifold of \tilde{X} and, in a neighborhood of every node, S coincides with P_0 in some local coordinates. Then $\pi^{-1}(S)$ is a real 3-dimensional submanifold with boundary of a resolution X , whose boundary is the union of the exceptional curves. Hence the exceptional curves satisfy a good relation and the nodes can be smoothed.

4.5. Topology change. Suppose that X and \tilde{X} are related by a conifold transition, i.e. they are respectively a resolution and a smoothing of a set of k nodes p_1, \dots, p_k of a conifold \tilde{X} . Let d be the rank of the group spanned by the homology classes of the exceptional curves $[C_j] \in H_2(X, \mathbb{Z})$ and c be the rank of the group spanned by the homology classes of the vanishing cycles $[S_j] \in H_3(\tilde{X}, \mathbb{Z})$. Then we have the following formulas relating the Betti numbers of X and \tilde{X} :

$$(19) \quad \begin{aligned} k &= d + c, \\ b_2(\tilde{X}) &= b_2(X) - d, \\ b_3(\tilde{X}) &= b_3(X) + 2c. \end{aligned}$$

For a proof we refer to the survey [25] and the references therein.

4.6. Conifold transitions and mirror symmetry. In [24] Morrison conjectures that given X and \tilde{X} two Calabi-Yau's related by a conifold transition, then their mirror manifolds (if they exist) are also related by a conifold transition, but in the reverse direction, i.e. the mirror of the resolution X is a smoothing \tilde{Y} and the mirror of the smoothing \tilde{X} is a resolution Y :

$$\begin{array}{ccc} X & \xrightarrow{CT} & \tilde{X} \\ MS \downarrow & & \downarrow MS \\ \tilde{Y} & \xleftarrow{CT} & Y \end{array}$$

Morrison in fact extends the conjecture to more general “extremal transitions” and clarifies that the conjecture makes sense only if the singular space \tilde{X} is “sufficiently” close to a large complex structure limit point of the complex moduli of \tilde{X} . He supports the conjecture with many examples and many others have appeared in the literature, e.g. in [2]. At the time Morrison wrote the article, the SYZ interpretation of mirror symmetry as dual special Lagrangian fibrations had just been proposed and, in the final remarks he suggests that “such an understanding could ultimately lead to a proof of the conjecture using the new geometric definition of mirror symmetry”. To achieve such a goal he suggests the “important and challenging problem to understand how such fibrations behave under an extremal transition”. The goal of our article is to take up this challenge, at least in the case of conifold transitions, and propose a general strategy using the techniques of the Gross-Siebert program and the properties of dual torus fibrations as studied in [7] and [4]. We will show that the strategy works in many special cases but we believe that a more general

statement should be in reach of current technologies. We also point out that in [26], Ruan sketches a Lagrangian fibration on one of the examples of [2] constructed via a conifold transition.

5. EXPLICIT FIBRATIONS ON THE LOCAL CONIFOLD

In this section we discuss explicit torus fibrations on the local conifold X_0 , on its smoothing Y_ϵ and on its small resolution X . These are slightly modified versions of Ruan's fibrations [26] (our maps are proper, Ruan's fibrations are not), see also [6]. The torus fibrations will be of two types: *positive* fibrations, which are T^2 invariant, and *negative* ones, which are S^1 invariant. Only positive fibrations will be Lagrangian, but in §6.5 we will sketch how to obtain Lagrangian versions of all fibrations on the resolution and on the smoothing. We also will not worry too much if the fibrations are not defined on the entire spaces, it is enough that they are well defined torus fibrations on a neighbourhood of the exceptional \mathbb{P}^1 or vanishing cycle.

5.1. Positive fibrations. We assume that the symplectic form on X is given by (14). Consider the T^2 -action on X which, for $\xi = (\xi_1, \xi_2) \in T^2$, is given by

$$(20) \quad \xi \cdot (z_1, z_2, z_3, z_4, t) = (\xi_1 z_1, \xi_1^{-1} z_2, \xi_2 z_3, \xi_2^{-1} z_4, \xi_1 \xi_2^{-1} t).$$

If we ignore the variable t , the same expression also gives a T^2 action on the conifold X_0 and on the smoothing Y_ϵ .

Example 5.1 (Resolution). We define the fibration on X and X_0 . The above action is Hamiltonian and the corresponding moment map is $\mu = (\mu_1, \mu_2)$ where:

$$\begin{aligned} \mu_1 &= |z_1|^2 - |z_2|^2 - \frac{\delta}{1 + |t|^2} \\ \mu_2 &= |z_3|^2 - |z_4|^2 + \frac{\delta}{1 + |t|^2}. \end{aligned}$$

Define $f_\delta : X \rightarrow \mathbb{R}^3$ by

$$(21) \quad f_\delta(z, t) = (\log |z_1 z_2 + z_3 z_4 - 1|, \mu_1(z, t), \mu_2(z, t)).$$

The locus $\text{Crit}(f_\delta) \subset X$ consists of 5 components:

$$(22) \quad \begin{aligned} L_0 &= \{(z, t) \in X \mid z = 0\} \cong \mathbb{CP}^1 \\ L_j &= \{(z, t) \in X \mid z_i = 0, i \neq j\} \cong \mathbb{C}, \quad j = 1, \dots, 4. \end{aligned}$$

These components are mapped to $\Delta \subset \{0\} \times \mathbb{R}^2$, where Δ consists of 5 edges forming 2 trivalent vertices as in Figure 5 (a).

The component $L_0 \cong \mathbb{P}^1$ is the exceptional locus of the resolution. One can check that $f_\delta(L_0)$ is the bounded edge of Δ , joining the two vertices, and $f_\delta(L_j)$ are the unbounded edges, $j = 1, \dots, 4$. One can check easily that f_δ is a Lagrangian 3-torus fibration. The singular fibres are of generic type along the edges of Δ , whereas over the vertices the fibres are of positive type. Observe that when $\delta = 0$, we obtain a torus fibration on the conifold X_0 , where the bounded edge $f_\delta(L_0)$ collapses to a point and Δ becomes a 4-valent graph.

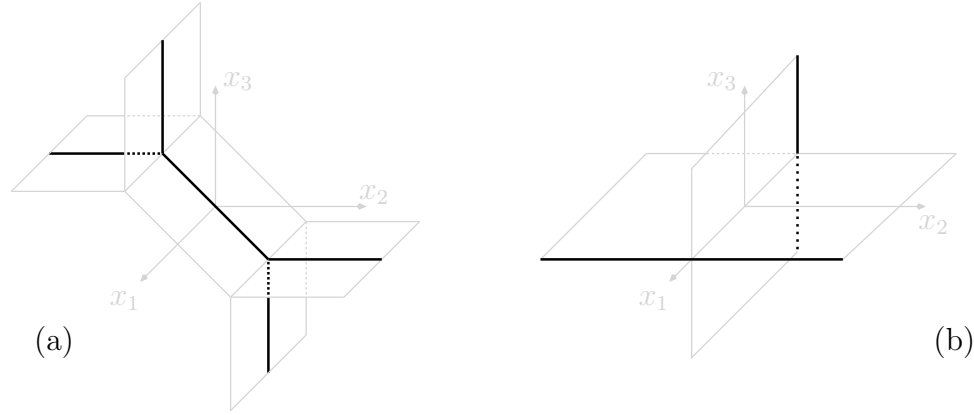


FIGURE 5. Positive fibrations: resolution (a), smoothing (b)

Example 5.2 (Smoothing). We now define the fibration on Y_ϵ . Let $f : Y_\epsilon \rightarrow \mathbb{R}^3$ be defined by

$$(23) \quad f(z) = (\log |z_1 z_2 + z_3 z_4 - 1|, |z_1|^2 - |z_2|^2, |z_3|^2 - |z_4|^2),$$

where the last two function components give the moment map of the torus action (20). One can check that f is a Lagrangian 3-torus fibration. The critical locus of f has two connected components:

$$\begin{aligned} C_1 &= \{z_1 = z_2 = 0, z_3 z_4 = -\epsilon\} \cong \mathbb{C}^* \\ C_2 &= \{z_3 = z_4 = 0, z_1 z_2 = \epsilon\} \cong \mathbb{C}^*. \end{aligned}$$

The discriminant locus of f has two disjoint components:

$$\begin{aligned} f(C_1) &= \{x_1 = \log |1 + \epsilon|, x_2 = 0\} \cong \mathbb{R}, \\ f(C_2) &= \{x_1 = \log |1 - \epsilon|, x_3 = 0\} \cong \mathbb{R}, \end{aligned}$$

depicted in Figure 5 (b). The singular fibres are all of generic type. The set:

$$(24) \quad Y_\epsilon \cap \{z_2 = \bar{z}_1, z_3 = -\bar{z}_4\} \cong S^3$$

is a Lagrangian cycle, vanishing as $\epsilon \rightarrow 0$. It is mapped by f to the set:

$$\{x_1 = \log |\epsilon - 2|z_3|^2 - 1|, x_2 = x_3 = 0, 0 \leq |z_3|^2 \leq \epsilon\}$$

which is a segment joining the components of the discriminant locus.

5.2. Negative fibrations. Consider the S^1 -action on X defined as follows:

$$(25) \quad \xi \cdot (z_1, z_2, z_3, z_4, t) = (\xi z_1, \xi^{-1} z_2, z_3, z_4, \xi t)$$

for $\xi \in S^1$. Ignoring t , the same formula also gives an S^1 action on X_0 and Y_ϵ .

Example 5.3 (Resolution). We give another fibration on X and on X_0 . The moment map of the action (25) on X is

$$\mu = |z_1|^2 - |z_2|^2 + \frac{\delta}{1 + |t|^2}.$$

The map $f_\delta : X \rightarrow \mathbb{R}^3$ is defined by

$$(26) \quad f_\delta(z, t) = (\mu(z, t), \log |z_3 - 1|, \log |z_4 - 1|)$$

defines a 3-torus fibration, with critical locus $\text{Crit}(f_\delta)$ consisting of two components:

$$(27) \quad D_i = \{z_j = 0, j \neq i\}, \quad i = 3, 4$$

These are mapped by f_δ onto the discriminant locus, which is the union of two lines as in Figure 6 (a):

$$(28) \quad \begin{aligned} f_\delta(D_3) &= \{x_1 = \delta, x_3 = 0\} \cong \mathbb{R} \\ f_\delta(D_4) &= \{x_1 = 0, x_2 = 0\} \cong \mathbb{R}. \end{aligned}$$

The singular fibres are all of generic type. A direct calculation shows that the

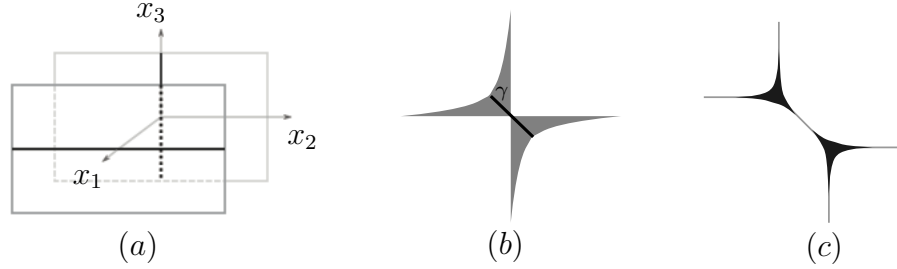


FIGURE 6. Negative fibrations: resolution (a), smoothing (b) and (c)

exceptional \mathbb{CP}^1 is mapped to the segment $\{x_2 = x_3 = 0, 0 \leq x_1 \leq \delta\}$ joining the components of (28). The fibres over p in the interior of this segment intersect the exceptional curve along an S^1 , which collapses to a point as p approaches either component of the discriminant. Observe that this fibration is not Lagrangian with respect to the symplectic form (14). We will discuss how to obtain Lagrangian fibrations of this model in §6.5. Notice that as $\delta \rightarrow 0$, the lines (28) come together to form a 4-valent vertex. We thus obtain a different fibration on the conifold X_0 .

Example 5.4 (Smoothing). We describe a fibration on Y_ϵ . The moment map for the action (25) on Y_ϵ is $\mu = |z_1|^2 - |z_2|^2$. The map $f : Y_\epsilon \rightarrow \mathbb{R}^3$ is

$$(29) \quad f(z) = (|z_1|^2 - |z_2|^2, \log |z_3 - 1|, \log |z_4 - 1|).$$

It is a smooth torus fibration but it is not Lagrangian with respect to the standard symplectic form. The critical locus is

$$\text{Crit}(f) = \{z_1 = z_2 = 0, z_3 z_4 = -\epsilon\}$$

and the discriminant $\mathcal{A} = f(\text{Crit}(f))$ has the shape of a 4-legged amoeba contained in the plane $\{x_1 = 0\} \subset \mathbb{R}^3$ as in Figure 6 (b). Specifically, if $\text{Log}(u, v) = (\log |u|, \log |v|)$ and

$$(30) \quad V = \{(u, v) \in (\mathbb{C}^*)^2 \mid (u + 1)(v + 1) + \epsilon = 0\},$$

then $\mathcal{A} = \text{Log}(V)$. A construction of a Lagrangian fibration on the smoothing will be explained in §6.5.

We now give a topological description of these fibrations in the spirit of Example 3.3 and [7]. Let

$$Y = T^2 \times \mathbb{R}^3$$

Define the following subsets of \mathbb{R}^3 : $\Delta_2 = \{(t, 0, 0) \mid t \in \mathbb{R}\}$ and for some $\delta \geq 0$, $\Delta_3 = \{(0, t, \delta) \mid t \in \mathbb{R}\}$. Let $\Delta = \Delta_2 \cup \Delta_3$. Fix a basis e_2, e_3 for $H_1(T^2, \mathbb{Z})$. Let $S^1 \subseteq T^2$ be a circle representing the class e_2 , then define $\Sigma_2 \subset Y$ to be the cylinder $S^1 \times \Delta_2$. If $S^1 \subseteq T^2$ represents e_3 , then define $\Sigma_3 = S^1 \times \Delta_3$. Then let $\Sigma = \Sigma_2 \cup \Sigma_3$. Now consider a space X and a map $\pi : X \rightarrow Y$ such that Y is the quotient of X with respect to an S^1 action on X and such that $\Sigma \subset Y$ is the image of the fixed point sets of the S^1 action. Moreover $X' = \pi^{-1}(Y - \Sigma)$ is a principal S^1 -bundle with $c_1 = \pm 1$. Then, as in Example 3.3, consider the trivial T^2 fibration $P : Y \rightarrow \mathbb{R}^3$ given by projection. The composition $f = P \circ \pi$ is a 3-torus fibration which is topologically equivalent to the one described in Example 5.3. Observe that when $\delta = 0$ the two surfaces Σ_2 and Σ_3 intersect transversely in one point, this implies that X becomes singular, and in fact it becomes homeomorphic to the conifold X_0 . When $\delta > 0$, X is homeomorphic to a small resolution of X_0 . Clearly the discriminant of this fibration is Δ and when $\delta > 0$ all fibres are of generic-singular type. When $\delta = 0$, Δ becomes “X” shaped and we have a more singular fibre over the origin (the vertex of Δ). If X_b is a smooth fibre of f take as a basis of $H_1(X_b, \mathbb{Z})$, $e_1(b), e_2(b), e_3(b)$, where e_2 and e_3 are the 1-cycles in $P^{-1}(b) = T^2$ chosen as above and e_1 is a fibre of the S^1 -bundle. In this basis, the monodromies around Δ_2 and Δ_3 are respectively given by matrices

$$(31) \quad T_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad T_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In particular, when $\delta = 0$, two opposite legs of Δ emanating from its vertex have the same monodromy.

We can also give a topological description of the fibration on the smoothing (Example 5.4). This can be done as follows. Take $\Delta_1 = \{(-t, 0, 0), t \geq 0\}$, $\Delta_2 = \{(0, -t, 0), t \geq 0\}$, $\Delta_3 = \{(t, t, 0), 0 \leq t \leq 1\}$, $\Delta_4 = \{(1+t, 1, 0), t \geq 0\}$, $\Delta_5 = \{(1, 1+t, 0), t \geq 0\}$ and define $\Delta = \cup_{i=1}^5 \Delta_i$. Then as above, we can construct a surface Σ in $Y = T^2 \times \mathbb{R}^3$ such that $P(\Sigma) = \Delta$, in such a way that for a point b in the interior of Δ_i we have that $P^{-1}(b) \cap \Sigma$ is circle in T^2 representing the class e_2 if $i = 1$ or 4 , the class e_3 if $i = 2$ or 5 and the class $-e_2 - e_3$ if $i = 3$. Then the construction of $f : X \rightarrow \mathbb{R}^3$ proceeds as above. In this case X is homeomorphic to a dense open subset of the smoothing Y_ϵ . Notice that in this case Δ has two vertices over which the fibration is of negative type.

Remark 5.5. Observe that the torus fibrations on the resolution given in Example 5.1 and the one on the smoothing given in Example 5.4 are topologically

“mirror dual” in the sense of [7]. The same is true for the torus fibrations on the smoothing given in Example 5.2 and on the resolution given in Example 5.3. This was already observed by Ruan [26] and Gross [6]. In the next section we will show that this is true at the level of “tropical manifolds”, where mirror duality is meant in the sense of discrete Legendre transform.

6. AFFINE GEOMETRY AND THE LOCAL TROPICAL CONIFOLD

Here we construct two *tropical* models of the conifold X_0 , which we call positive and negative nodes. We show that these two models are mirror to each other in the sense of the discrete Legendre transform. Moreover they give rise to torus fibrations on the conifold X_0 which are topologically equivalent to the ones in Examples 5.1 and 5.3 respectively. Then we introduce tropical resolutions and smoothings of these nodes and show that discrete Legendre transform exchanges smoothings with resolutions. Finally we explain how to obtain Lagrangian 3-torus fibrations from these tropical manifolds.

6.1. Tropical nodes.

Example 6.1. (Negative node) Let

$$(32) \quad T = \text{Conv}\{(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 0, 1), (0, 1, 1)\}$$

be a triangular prism in \mathbb{R}^3 . Construct a tropical manifold $(\check{B}, \check{\mathcal{P}}, \check{\phi})$ as follows. Take two copies of T and choose in each copy a square face, then label the vertices of these square faces by v_1, v_2, v_3 and v_4 as in Figure 7. Now glue the two copies of T along the chosen faces by the unique affine linear transformation which matches the vertices with the same label. This is \check{B} . We denote by \check{e} the common face of the two copies of T . In \mathbb{R}^3 , let e_1, e_2, e_3 denote the standard basis and let (x, y, z) be the coordinates. Consider the fan Σ in \mathbb{R}^3 whose 3-dimensional cones are two adjacent octants, i.e. $\text{Cone}(e_1, e_2, e_3)$, and $\text{Cone}(e_1, -e_2, e_3)$. At each vertex v_j choose a fan structure which maps the tangent wedge at v_j of the first copy of T to the first octant and the tangent wedge of the second copy of T to the second octant (see Figure 7). Then the discriminant locus Δ is the union of the segments joining the barycenter of \check{e} to the barycenters of its edges. We can easily compute the monodromy of $T\check{B}_0$. In

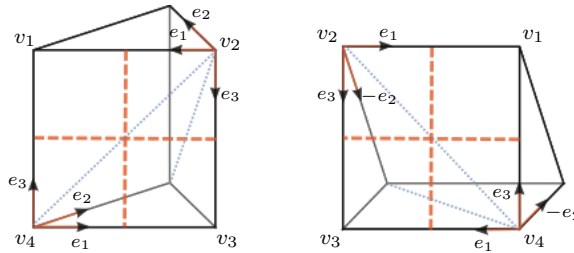


FIGURE 7. Match the vertices with the same labels. The arrows show the fan structure at the vertices. The (red) dashed lines denote Δ .

fact, start at the vertex v_4 and choose the vectors $\{e_1, e_2, e_3\}$ mentioned above as a basis for $T_{v_4}\check{B}_0$. Then consider a path which goes into the first prism, passes through v_3 and then comes back to v_4 going into the second prism. Monodromy along this path is given by the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Similarly, consider a path which goes into the first prism, passes through v_1 and then comes back to v_4 going into the second prism. Monodromy along this path is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Observe that $T_{v_4}\check{B}_0$ has a 2-dimensional subspace, spanned by e_1 and e_3 which is invariant with respect to both monodromy transformations. Now define the strictly convex MPL-function $\check{\phi}$ so that on the fan Σ it is given by

$$\check{\phi}(x, y, z) = \begin{cases} y & y \geq 0, \\ 0 & y \leq 0. \end{cases}$$

If we apply the discrete Legendre transform to $(\check{B}, \check{\mathcal{P}}, \check{\phi})$, we obtain the second tropical model for the conifold.

Example 6.2. (Positive node) The triple (B, \mathcal{P}, ϕ) , mirror to $(\check{B}, \check{\mathcal{P}}, \check{\phi})$ above, can be described as follows. The manifold B can be identified with $\mathbb{R}^2 \times [0, 1]$. If we denote by $Q_j \subseteq \mathbb{R}^2$, $j = 1, 2, 3, 4$ the four closed quadrants of \mathbb{R}^2 , i.e. $\text{Cone}((1, 0), (0, 1))$, $\text{Cone}((-1, 0), (0, 1))$, $\text{Cone}((-1, 0), (0, -1))$ and $\text{Cone}((1, 0), (0, -1))$ respectively, then the 3-dimensional polytopes of \mathcal{P} are $L_j = Q_j \times [0, 1]$. In fact L_j is dual to the vertex v_j in $\check{\mathcal{P}}$. Here we ignore the polytopes dual to vertices not contained in \check{e} . The two vertices $p_0 = (0, 0, 0)$ and $p_1 = (0, 0, 1)$ are dual to the two triangular prisms. Consider the fan Σ whose 3-dimensional cones are:

$$\text{Cone}(e_1, -e_1 - e_2, e_3), \text{Cone}(e_1, e_2, e_3), \text{Cone}(e_1, e_2, -e_3), \text{Cone}(e_1, -e_1 - e_2, -e_3).$$

The fan structure at p_0 maps tangent wedges of the polytopes L_1, L_2, L_3 and L_4 respectively to the first, second, third and fourth cone above. Similarly at p_1 , the tangent wedges to L_1, L_2, L_3 and L_4 are mapped respectively to the first, fourth, third and second cone above, see Figure 8.

One easily checks that the discriminant locus is given by

$$\Delta = \{(t, 0, 1/2), t \in \mathbb{R}\} \cup \{(0, t, 1/2), t \in \mathbb{R}\},$$

with a 4-valent vertex in $(0, 0, 1/2)$. We can compute monodromy at p_0 , where we choose the basis $\{e_1, e_2, e_3\}$ of $T_{p_0}B_0$. Consider a path which goes from p_0

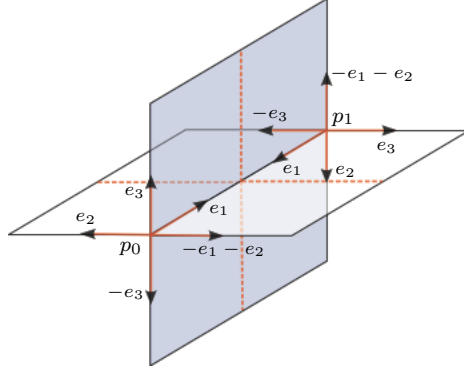


FIGURE 8. The tropical positive node. The arrows indicate the fan structure.

into L_2 , reaches p_1 and then comes back to p_0 passing into L_1 . Monodromy along this path is given by the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Similarly, consider a path which goes into L_2 , reaches p_1 and comes back to p_0 passing into L_3 . Monodromy along this path is given by the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Observe that $T_{p_0}B_0$ has a 1-dimensional subspace, spanned by e_1 which is invariant with respect to both monodromy transformations.

We can now give a more general notion of “tropical conifold”, which is a tropical manifold where Δ may have 4-valent vertices (called nodes) modeled on the previous examples.

Definition 6.3. We say that a 3-dimensional tropical manifold (B, \mathcal{P}, ϕ) is a *tropical conifold* if Δ has vertices of valency 3 or 4. The 3-valent vertices are either of positive or negative type (see Examples 2.4 and 2.3). Every 4-valent vertex has a neighborhood which is integral affine isomorphic to a neighborhood of the vertex of Δ either in Example 6.1 or Example 6.2. In the former case the vertex is called a negative node, in the latter a positive node.

Of course, the discrete Legendre transform of a tropical conifold is also a tropical conifold.

6.2. Local tropical resolutions and smoothings. We will describe two procedures which we claim should be the tropical analogue to resolving or smoothing a node. In fact each procedure uses discrete Legendre transform so that while it resolves the positive node (resp. negative) it smooths the mirror negative one (resp. positive).

Let us first describe the local resolution of a positive node, which simultaneously smoothes the negative one. A positive node is contained in the interior of an edge e of \mathcal{P} which belongs to four 3-dimensional faces. In the mirror $(\check{B}, \check{\mathcal{P}}, \check{\phi})$, the dual face to the edge e is the square face \check{e} . To “smooth” the negative node we proceed as follows. Subdivide \check{e} in two triangles by adding a diagonal. Then we need to find a suitable polyhedral decomposition $\check{\mathcal{P}}'$ which is a refinement of $\check{\mathcal{P}}$ and which induces the chosen subdivision of \check{e} . Next, we need an MPL-function $\check{\phi}'$ which is strictly convex with respect $\check{\mathcal{P}}'$. Notice that the given subdivision of \check{e} also implies a change in the discriminant locus (see Figure 9), where the 4-valent vertex splits in two 3-valent ones. This is a smoothing of the negative node. The resolution of the positive node is the discrete Legendre transform of the smoothing of the node. The process is illustrated in Figure 9.

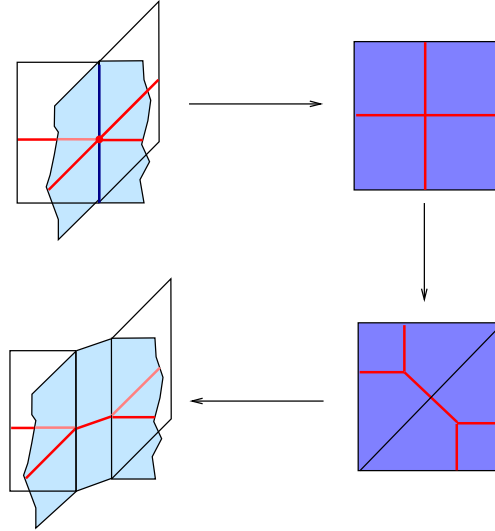


FIGURE 9. The resolution of a positive node: the horizontal arrows are the discrete Legendre transform, the vertical one is the smoothing of the negative node.

Similarly we can describe the resolution of a negative node with the simultaneous smoothing of its mirror positive one (see also Figure 10). To smooth a positive node, first subdivide the edge ℓ where it lies in two new edges. Then we need a refinement \mathcal{P}' of the decomposition \mathcal{P} which induces the given subdivision of the edge. This decomposition has an extra vertex, hence we need to define a suitable fan structure at this vertex. Next, we find a new MPL function ϕ' which is strictly convex with respect to \mathcal{P}' . The choice of the fan structure at the new vertex must have the effect of separating the two lines of discriminant locus. This is a smoothing of a positive node. The discrete Legendre transform gives us the resolution of the negative node. Note that the two lines of the discriminant locus in the negative node are separated by adding in between a new 3-dimensional polytope, mirror to the new vertex.

In the following paragraphs we apply this idea in detail.

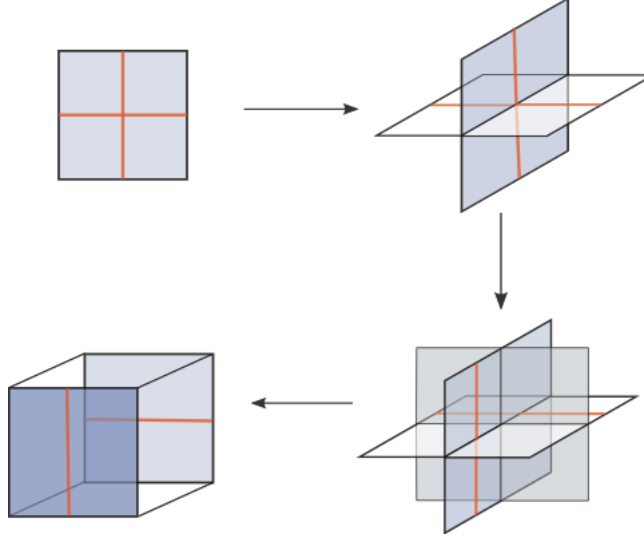


FIGURE 10. The resolution of a negative node.

6.3. Resolving a positive node and smoothing its mirror. In Example 6.1, let us subdivide \check{e} by taking the diagonal from v_2 to v_4 (see Figure 7). Assume that v_2 and v_4 correspond respectively to the vertices $(0, 1, 1)$ and $(1, 0, 0)$ in the first copy of T and to $(1, 0, 0)$ and $(0, 1, 1)$ in the second copy. Now subdivide T in the two polytopes $\text{Conv}\{(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 1, 1)\}$ and $\text{Conv}\{(0, 0, 0), (0, 0, 1), (1, 0, 1), (1, 0, 0), (0, 1, 1)\}$, and consider this subdivision for each copy of T in $\check{\mathcal{P}}$. This gives the new decomposition $\check{\mathcal{P}}'$. Let us now find the new function $\check{\phi}$. The decomposition induces also a decomposition of the fans at the vertices. For instance, at v_2 the new fan is Σ'_2 , whose 3-dimensional cones are:

$$\begin{aligned} &\text{Cone}(e_1, e_2, e_2 + e_3, e_1 + e_3), \text{Cone}(e_2 + e_3, e_1 + e_3, e_3), \\ &\text{Cone}(e_1, -e_2, e_1 + e_3), \text{Cone}(e_3, -e_2, e_1 + e_3). \end{aligned}$$

The first two subdivide the first octant and the other two subdivide the second one. Similarly at v_4 , the fan is Σ'_4 whose 3-dimensional cones are:

$$\begin{aligned} &\text{Cone}(e_1, e_2, e_1 + e_3), \text{Cone}(e_1 + e_3, e_2, e_3), \\ &\text{Cone}(e_1, e_1 - e_2, e_1 + e_3), \text{Cone}(e_3, -e_2, e_1 - e_2, e_1 + e_3). \end{aligned}$$

Now define an MPL-function $\check{\phi}$ as follows. It is the zero function on the fan at v_1 . At v_2 it is the unique piecewise linear function which is 1 on e_3 and zero on all other generators of one dimensional cones. Similarly at v_4 define $\check{\phi}$ to be 1 on e_1 and zero on all other generators. At v_3 , $\check{\phi}$ is -1 on e_2 and zero on all other generators. Now define the new function $\check{\phi}'$ by

$$\check{\phi}' = 2\check{\phi} + \check{\phi}.$$

We can verify that $\check{\phi}'$ is a well defined strictly convex MPL function on $(\check{B}, \check{\mathcal{P}}')$. Notice also that the discriminant locus Δ , with respect to this structure, has two negative vertices. Applying the discrete Legendre transform to $(\check{B}, \check{\mathcal{P}}', \check{\phi}')$ gives a new tropical manifold $(B', \mathcal{P}', \phi')$ which we define to be the resolution of a positive node (see also Figure 11). There are 4 polytopes in \mathcal{P}' corresponding

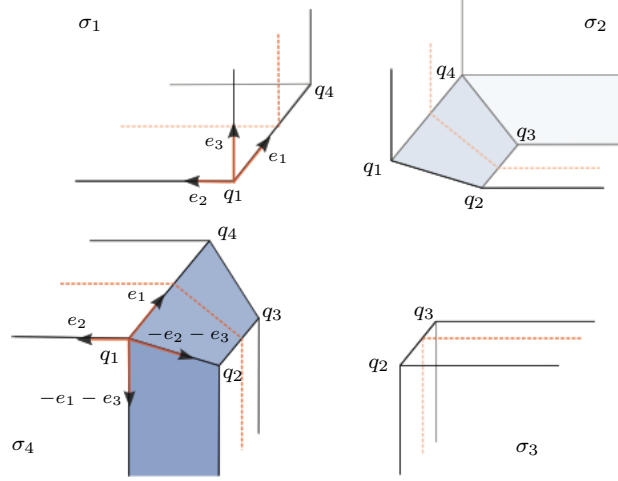


FIGURE 11.

to the vertices v_1, \dots, v_4 of $\check{\mathcal{P}}'$, which we denote respectively by $\sigma_1, \dots, \sigma_4$. We also denote by ℓ_{jk} the 2-dimensional face in \mathcal{P}' corresponding to the edge from v_j to v_k , if such an edge exists. The polytopes σ_2 and σ_4 are integral affine isomorphic to the subset in \mathbb{R}^3 given by the following inequalities

$$\begin{cases} -2 \leq y \leq 0, \\ x \geq -1, \\ z \geq 0, \\ x + z \geq 0, \\ x - y \geq 0. \end{cases}$$

They intersect along the face ℓ_{24} . We label the vertices of ℓ_{24} by q_1, q_2, q_3 and q_4 as in Figure 11. The edge from q_1 to q_4 is the intersection between σ_1 , σ_2 and σ_4 . The polytopes σ_1 and σ_3 are as in the picture. Now consider the fan Σ_1 in \mathbb{R}^3 whose 3-dimensional cones are:

$$\text{Cone}(e_1, e_2, e_3), \text{Cone}(e_1, e_3, -e_2 - e_3), \text{Cone}(e_1, e_2, -e_1 - e_3, -e_2 - e_3).$$

Notice that Σ_1 is part of the normal fan of the two polytopes in $\check{\mathcal{P}}'$ containing v_1, v_2 and v_4 . Thus the fan structure at q_1 maps the tangent wedge to σ_1 , σ_2 and σ_4 to the first, second and third cone respectively. Here e_1 and e_2 span a 2-dimensional cone corresponding to ℓ_{14} , e_1 and e_3 span a cone corresponding to ℓ_{12} and e_1 and $-e_2 - e_3$ span a cone corresponding to ℓ_{24} . Similarly, at the vertex q_4 the fan structure maps the tangent wedge to σ_1 , σ_4 and σ_2 to the first,

second and third cones of Σ_1 respectively. The tangent wedges to ℓ_{14} , ℓ_{12} and ℓ_{24} correspond respectively to $\text{Cone}(e_1, e_3)$, $\text{Cone}(e_1, e_2)$ and $\text{Cone}(e_1, -e_2 - e_3)$. Let Σ_2 be the fan whose 3-dimensional cones are:

$$\text{Cone}(e_1, e_2, e_3), \text{Cone}(e_1, e_3, -e_1 - e_2 - e_3), \text{Cone}(e_1, e_2, -e_1 - e_2 - e_3).$$

Notice that Σ_2 is part of the normal fan of the two polytopes in $\check{\mathcal{P}}'$ containing v_3, v_2 and v_4 . Hence at the vertices q_2 and q_3 the fan structure maps the tangent wedges to σ_3 , σ_4 and σ_2 respectively to the first, second and third cone of Σ_2 . The tangent wedges to ℓ_{23} , ℓ_{34} and ℓ_{24} correspond respectively to $\text{Cone}(e_1, e_2)$, $\text{Cone}(e_1, e_3)$ and $\text{Cone}(e_1, -e_1 - e_2 - e_3)$.

It can be verified that Δ has two 3-valent positive vertices (see Figure 11), one on the barycenter of the edge from q_1 to q_4 and the other on the barycenter of the edge from q_2 to q_3 . At q_1 the direction e_1 is the only direction which is invariant with respect to monodromy around all edges of Δ .

6.4. Resolving a negative node and smoothing its mirror. To smooth the positive node in Example 6.2, consider the following refinement \mathcal{P}' of \mathcal{P} . First rescale every polytope L_j by a factor of two, so that $L_j = Q_j \times [0, 2]$. Now subdivide each L_j in the polytopes $L_j^- = Q_j \times [0, 1]$ and $L_j^+ = Q_j \times [1, 2]$. We have thus added a new vertex which we denote by q . We now define the fan structure at q . Let Σ_q be the fan in \mathbb{R}^3 whose maximal cones are the eight octants, namely $\text{Cone}(\pm e_1, \pm e_2, \pm e_3)$. The fan structure at q then identifies the eight tangent wedges of L_j^\pm , $j = 1, \dots, 4$ with these octants in the obvious way. The fan structures at p_0 and p_1 is unchanged. The new discriminant locus consists of two disjoint lines of generic-singularities: $\Delta^- = \{(0, t, 1/2), t \in \mathbb{R}\}$ and $\Delta^+ = \{(t, 0, 3/2), t \in \mathbb{R}\}$. Assume that in the fan Σ_q , the vector e_1 is tangent to the edge from q to p_1 . On Σ_q consider the piecewise linear function $\tilde{\phi}$ which takes the value 1 on e_1 and the value 0 on all other generators of 1-dimensional cones of Σ_q . We have that

$$\phi' = \phi + \tilde{\phi}$$

is a well defined, strictly convex piecewise linear function on (B, \mathcal{P}') . This structure defines the tropical smoothing of the positive node. A resolution of the negative node is given by the discrete Legendre transform $(\check{B}, \check{\mathcal{P}}')$. This can be described as follows.

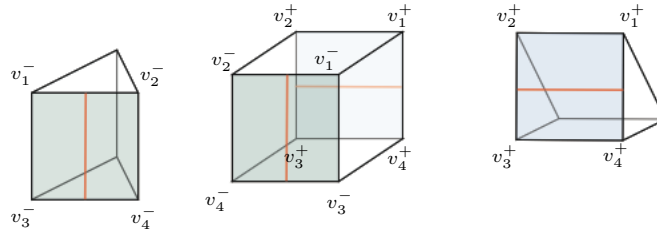


FIGURE 12.

Take two copies of the triangular prism T , defined in (32) and a cube $Q = [0, 1]^3 \subseteq \mathbb{R}^3$. Then glue the two copies of T onto two opposite faces of Q as in Figure 12, by matching vertices which have the same label. This is B with its polyhedral decomposition $\check{\mathcal{P}}'$. The cube Q is obviously mirror to the new vertex q in \mathcal{P}' . The fan structure at a vertex v_j^\pm is quite simple. In fact consider the fan Σ whose 3-dimensional cones are $\text{Cone}(e_1, e_2, e_3)$ and $\text{Cone}(e_1, -e_2, e_3)$. Then the fan structure identifies the tangent wedges of T and Q at v_j^\pm with these two octants. It is easy to see that Δ consists of two disjoint lines as in Figure 12. This is the tropical resolution of a negative node.

There is also an alternative version of the smoothing of a positive node which proceeds as follows. Rescale all polytopes L_j by a factor of 4 and subdivide each L_j in the polytopes $L_j^- = Q_j \times [0, 1]$ and $L_j^+ = Q_j \times [1, 4]$. Now consider the fan whose three dimensional cones are

$$\begin{aligned} &\text{Cone}(e_1, e_1 - e_2, e_3 - e_1), \text{Cone}(e_1, e_2, e_3 - e_1), \text{Cone}(e_1, e_2, -e_3), \\ &\text{Cone}(e_1, e_1 - e_2, -e_3), \text{Cone}(-e_1, e_1 - e_2, e_3 - e_1), \text{Cone}(-e_1, e_2, e_3 - e_1), \\ &\text{Cone}(-e_1, e_2, -e_3), \text{Cone}(-e_1, e_1 - e_2, -e_3). \end{aligned}$$

Now define the fan structure at the new vertex $q = (0, 0, 1)$ in such a way that the tangent wedges at q to the polytopes L_1^-, L_2^-, L_3^- and L_4^- are mapped respectively to the first, second, third and fourth cone, while the tangent wedges to L_1^+, L_2^+, L_3^+ and L_4^+ are mapped respectively to the fifth, sixth, seventh and eighth cone. It can be checked that with such a choice of subdivision and fan structure at q , the discriminant consists of the two components $\Delta^- = \{(t, 0, 1/2) \mid t \in \mathbb{R}\}$ and $\Delta^+ = \{(0, t, 5/2), t \in \mathbb{R}\}$. In particular, compared with the previous choices, the two lines of Δ have now moved in the opposite way. We leave it to the reader to determine a suitable strictly convex MPL function.

The two possibilities for the smoothing should correspond, in the mirror, to the two choices of small resolution.

6.5. Tropical nodes and Lagrangian fibrations. In Examples 5.1 and 5.2 we have described Lagrangian fibrations defined over (dense open subsets of) the local conifold X_0 , of a small resolution X and of a smoothing Y_ϵ . Using the results of [3] and [4] one can show that the affine structure induced on $\mathbb{R}^3 - \Delta$ by the action coordinates is affine isomorphic respectively to the affine structures given in Examples 6.2 for the tropical positive node, in §6.3 for its resolution and in §6.4 for its smoothing. This shows that we can think of X_0 , X and Y_ϵ as a (partial) symplectic compactification X_B of the symplectic manifold $X_{B_0} = T^*B_0/\Lambda^*$ constructed from the relevant tropical manifold (B, \mathcal{P}, Δ) (see diagram (2)).

In the case of Examples 5.3 and 5.4 the given fibrations on X_0 , X and Y_ϵ are not Lagrangian. In any case, one can show that the union of smooth fibres gives torus bundles which are topologically equivalent to the torus bundle X_{B_0} where (B, \mathcal{P}, Δ) is respectively the tropical manifold of Example 6.1 for the negative node, of §6.4 for its resolution and of §6.3 for its smoothing. This can be seen by simply comparing monodromy transformations, which are the only topological

invariants. So, at least at a topological level, we can consider X_0 , X and Y_ϵ as the compactification of X_{B_0} . Notice that the resolution and smoothing of the tropical negative node have symplectic compactifications, which exists by the result of [4], but we do not know if what we obtain is symplectomorphic to X or Y_ϵ , although we strongly believe this is true.

From these observations it follows that given a tropical conifold (B, \mathcal{P}) as in Definition 6.3, we can always construct its topological compactification X_B , which will be homeomorphic to a space with conifold singularities, in correspondence with the positive or negative nodes. We also believe one can put a symplectic structure on X_B , extending the one on X_{B_0} , which makes X_B into a symplectic conifold in the sense of [28]. In fact, from the above observations, this is true if B does not have negative nodes. We will call X_B the conifold associated to B . Notice also that, as in the smooth case (see Remark 3.6), we could consider $\check{X}_{B_0} = TB_0/\Lambda$. Then \check{X}_{B_0} can be compactified to form \check{X}_B . Over positive (resp. negative) nodes we glue the negative (resp. positive) fibration of the local conifold and similarly with positive and negative vertices. We still have that \check{X}_B is homeomorphic to $X_{\check{B}}$.

It is also reasonable to expect that the Gross-Siebert theorem ([12]), which associates to a tropical manifold a toric degeneration of Calabi-Yau manifolds, can be extended to tropical conifolds. Namely, one can expect that given a tropical conifold, we can reconstruct a toric degeneration whose fibres are Calabi-Yau manifolds with nodes.

7. GOOD RELATIONS

We now start addressing the question of when a given set of nodes in a tropical conifold B can be simultaneously resolved/smoothed. We introduce the notion of a tropical 2-cycle and we prove that if a set of nodes is contained in a tropical 2-cycle, then both the vanishing cycles inside a smoothing of X_B and the exceptional curves inside a small resolution of the mirror $X_{\check{B}}$ satisfy a good relation. Hence, morally, the obstructions to the symplectic resolution of X_B and to the complex smoothing of the mirror $X_{\check{B}}$ vanish simultaneously. In this section we assume that the tropical conifold B is oriented. In particular this implies that monodromy $\tilde{\rho}$ has values in $\mathrm{Sl}(\mathbb{Z}, n)$ and that the fibres of X_B have a canonical orientation.

7.1. Tropical 2-cycles. Recall that the tropical hyperplane V^{n-1} in \mathbb{R}^n is the set of points in \mathbb{R}^n where the following piecewise linear function fails to be smooth

$$f(x_1, \dots, x_n) = \max\{x_1, \dots, x_n, 0\}.$$

The point $(0, \dots, 0)$ is called the vertex of the tropical hyperplane. We call V^1 and V^2 the tropical line and plane respectively.

Our definition of a tropical 2-cycle resembles the definition of a tropical surface such as in [22], but it has a more topological flavor. A tropical 2-cycle will be a map from a certain space S to B plus some other data.

Definition 7.1. A *tropical domain* S is a compact Hausdorff topological space such that for every $p \in S$ there is a neighborhood U of p and a homeomorphism $\phi : (U, p) \rightarrow (W, q)$, where (W, q) can be one of the following pairs (see Figure 13):

- a) $q \in \mathbb{R}^2$ and W is a neighborhood of q (p is called a *smooth point* of S);
- b) q is the vertex of a tropical plane V^2 and W is a neighborhood of q in V^2 (p is called an *interior vertex* of S);
- c) $q = (0, 0, 0) \in V^1 \times \mathbb{R}$ and W is a neighborhood of q in $V^1 \times \mathbb{R}$ (p is called an *interior edge point*);
- d) W is a neighborhood of $q = (0, 0)$ in the closed half plane $\{x \geq 0\} \subset \mathbb{R}^2$ (p is called a *smooth boundary point*);
- e) $q = (0, 0, 0) \in V^1 \times \mathbb{R}_{\geq 0}$ and W is a neighborhood of q (p is called a *boundary vertex*);

We call points of type a), b), c) interior points. Points of type d) and e) form the *boundary* of S , which we denote by ∂S . We also denote the set of smooth points by S_{sm} . We will also assume that all connected components of S_{sm} are orientable and we fix an orientation. The union of all interior edge points forms a 1-dimensional manifold, whose connected components we call the *interior edges* of S .

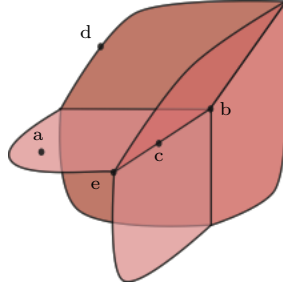


FIGURE 13. A tropical domain with smooth points (a), interior vertex (b), interior edge points (c), smooth boundary points (d) and boundary vertices (e).

We can now give the definition of a tropical 2-cycle:

Definition 7.2. Let (B, \mathcal{P}, ϕ) be a tropical conifold. A *tropical 2-cycle* in B is the data (S, j, v) where

- a) S is a tropical domain and $j : (S, \partial S) \rightarrow (B, \Delta)$ is an embedding;
- b) $j^{-1}(\Delta) = \partial S \cup \{q_1, \dots, q_r\}$, where q_1, \dots, q_r are smooth points and $j(q_k)$ is an edge point of Δ for all $k = 1, \dots, r$. We denote

$$S_0 = S_{sm} - \{q_1, \dots, q_r\};$$

- c) v is a primitive, integral, parallel vector field defined along $j(S_0)$;
- d) $j(p)$ is a negative vertex of Δ if and only if p is a boundary vertex of S ;

Notice that v induces a rank 2 subvector bundle \mathcal{F} of T^*B_0 over $j(S_0)$, where:

$$\mathcal{F}_q = \ker v(q) = \{\alpha \in T_q^*B_0 \mid \alpha(v(q)) = 0\}.$$

for every $q \in j(S_0)$. The above properties imply that if $j(p)$ is a node or a positive vertex, then p is a smooth boundary point. We may consider ∂S as a graph where boundary vertices are trivalent vertices and p is a bivalent vertex if and only if $j(p)$ is a node or a positive vertex. Then edges of ∂S are mapped to a subset of the edges of Δ . We require in addition the following properties (see also Figure 14):

- e) if $j(p)$ is a negative node then the two edges of ∂S emanating from p are mapped to edges of Δ as in Figure 14, picture (4);
- f) if $j(p)$ is a positive node then the two edges of ∂S emanating from p are mapped to edges of Δ as in Figure 14, picture (3);
- g) let $p \in S$ be an interior edge point lying on the edge e . Choose an orientation of e . Given a small connected neighborhood U of p , $j(U \cap S_{sm})$ has 3 connected components. The orientation of e and the orientation of B induce a cyclic ordering of these components. Denote by v_1, v_2, v_3 the vector field v restricted to these components, indexed according to the ordering. Then the v_k 's span a rank two subspace of $T_p B$ and they satisfy the following *balancing condition*

$$\epsilon_1 v_1 + \epsilon_2 v_2 + \epsilon_3 v_3 = 0,$$

where $\epsilon_k = 1$ if the chosen orientation of e coincides with the orientation induced from the orientation of the k -th component of $j(U \cap S_{sm})$, otherwise $\epsilon_k = -1$.

- h) if e is an edge of ∂S and U a small neighborhood of $j(e)$, then for all points $q \in j(S_{sm}) \cap U$, \mathcal{F}_q coincides with the monodromy invariant subspace with respect to monodromy around $j(e)$;
- i) if q_k is one of the smooth points such that $j(q_k)$ is an edge point of Δ , then monodromy of \mathcal{F} around $j(q_k)$ is conjugate to the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix};$$

- j) if p is an interior vertex point, and U is a small connected neighborhood of p , then $j(U \cap S_{sm})$ has 6 connected components. Let v_k , $k = 1, \dots, 6$ denote the restrictions of v to these components. Then the v_k 's span the whole of $T_p B$.

This definition differs from more common definitions of tropical varieties appearing in the literature. Here the embedding j is topological, i.e. the components of S_{sm} do not have to be mapped as affine subspaces of B . The more rigid tropical nature is captured by the vector field v , which has to be parallel along the components of S_{sm} and has to satisfy the balancing condition. In particular the boundary of S can also be mapped to a “curved” Δ (see §2.9 and Remark 3.5).

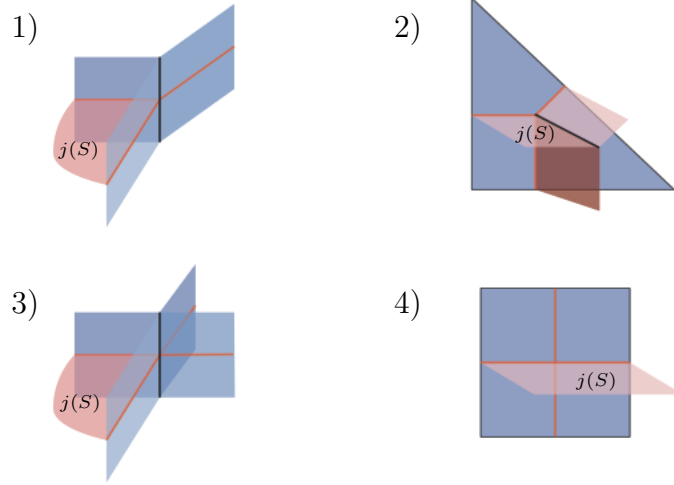


FIGURE 14. How $j(S)$ interacts with Δ : 1) at a positive vertex; 2) at a negative vertex; 3) at a positive node; 4) at a negative node.

Given a tropical conifold (B, \mathcal{P}, ϕ) , it follows from §6.5 that we can reconstruct the topological conifold X_B . Strictly speaking, we do not have general symplectic or complex reconstruction theorems for conifolds, nevertheless we have topological smoothings and resolutions of X_B and therefore we can speak about vanishing cycles and exceptional curves and it makes sense to ask whether these satisfy good relations. We will prove the following:

Theorem 7.3. Let (B, \mathcal{P}, ϕ) be an oriented tropical conifold and $(\check{B}, \check{\mathcal{P}}, \check{\phi})$ its Legendre dual. Let (S, j, v) be a tropical 2-cycle and p_1, \dots, p_k be the points of S which are mapped to nodes of B . Then the vanishing cycles L_1, \dots, L_k in a smoothing of X_B and the exceptional curves C_1, \dots, C_k in a small resolution of $X_{\check{B}}$ satisfy a good relation.

We want to construct a 4-dimensional submanifold with boundary \tilde{S} inside a smoothing of X_B and a 3-dimensional submanifold with boundary \tilde{S}^* inside a resolution of $X_{\check{B}}$ such that $\partial\tilde{S} = \cup_{i=1}^k L_i$ and $\partial\tilde{S}^* = \cup_{i=1}^k C_i$. Equivalently (see §4.4) we may think of \tilde{S} as a 4-dimensional submanifold without boundary inside X_B , containing the nodes, such that in local coordinates near the nodes, \tilde{S} coincides with the local collar Σ_0 as in (17). Similarly \tilde{S}^* may be thought as a subset of $X_{\check{B}}$, which, away from the nodes, is a submanifold and in local coordinates near the nodes, it coincides with the local collar P_0 as in (18). The idea is as follows. Consider the rank two bundle \mathcal{F} defined by v . Since v is integral, we have that $\mathcal{F}/(\mathcal{F} \cap \Lambda^*)$ defines a 2-torus bundle over $j(S_0)$. For every $\alpha \in \mathcal{F}_q$, denote by $[\alpha]$ its class inside $\mathcal{F}_q/(\mathcal{F}_q \cap \Lambda^*)$. Given a section $\sigma : B_0 \rightarrow X_{B_0}$, define

$$(33) \quad \tilde{S}_0 = \{(q, \sigma(q) + [\alpha]) \mid q \in j(S_0) \text{ and } \alpha \in \mathcal{F}_q\}.$$

Essentially \tilde{S}_0 is $\mathcal{F}/(\mathcal{F} \cap \Lambda^*)$ translated by σ . We want to prove that \tilde{S}_0 can be compactified (perhaps after a small deformation) to form the submanifold \tilde{S} such that $f(\tilde{S}) = j(S)$.

The construction of \tilde{S}^* is similar. First of all, as observed in Remark 3.6, instead of working in $X_{\tilde{B}}$ we can equivalently work in \check{X}_B , which is formed using the tangent bundle of B_0 . We denote by $\check{f} : \check{X}_B \rightarrow B$ the fibration. The vector field v generates a 1-dimensional subbundle of TB_0 along $j(S_0)$, which we call \mathcal{V} . Then $\mathcal{V}/\mathcal{V} \cap \Lambda$ is an S^1 bundle over $j(S_0)$. We form \tilde{S}_0^* by translating $\mathcal{V}/\mathcal{V} \cap \Lambda$ by a suitable section $\check{\sigma} : B \rightarrow \check{X}_B$. The set \tilde{S}^* is constructed as a suitable compactification of \tilde{S}_0^* .

As we will see in the construction, σ and $\check{\sigma}$ should not be genuine sections. They should be maps from $j(S)$ to X_B (resp. \check{X}_B) which are sections restricted to $j(S_0)$ but such that $\sigma(j(\partial S)) \subset \text{Crit } f$ (resp. $\check{\sigma}(j(\partial S)) \subset \text{Crit } \check{f}$). We can assume that σ and $\check{\sigma}$ both coincide with the zero section away from $j(\partial S)$ (see Remark 3.5). Let us now study local models in some detail.

7.2. Local models at smooth boundary points.... We discuss local models for \tilde{S} and \tilde{S}^* near an edge $e \subseteq j(\partial S)$ of Δ . In this case, the local model for both torus fibrations f and \check{f} is the generic singular fibration of Example 3.2 (see also the explicit map in (6)). Condition (h) of Definition 7.2 implies that v and \mathcal{F} are uniquely determined in a neighborhood of e , so that the lifts \tilde{S}_0 and \tilde{S}_0^* only depend on the choice of sections.

Notice that v is the unique (up to sign) primitive integral vector such that the monodromy T around the edge e satisfies $T(w) - w = mv$ for all $w \in T_b B_0$, where $m \in \mathbb{Z}$ depends on w . It then follows from the first part of Example 3.2, that for every $b \in B_0$ near e , the circle $\mathcal{V}_b/\mathcal{V}_b \cap \Lambda$ must belong to the same homology class as the orbit of the S^1 action (the cycle called e_1). In the explicit model (6), this class is the orbit of the action $\xi \cdot (z_1, z_2, z_3) = (\xi z_1, \xi^{-1} z_2, z_3)$ for $\xi \in S^1$. Recall that the fixed points of this action coincide precisely with $\text{Crit } f$. Suppose that U is a small neighborhood of e and let $\check{\sigma} : j(S) \cap U \rightarrow \check{X}_B$ be a section such that $\check{\sigma}(e) \subseteq \text{Crit } \check{f}$. Then we can define $\tilde{S}^* \cap \check{f}^{-1}(U)$ to be the union of the S^1 -orbits of the points $\check{\sigma}(p)$ for all $p \in j(S) \cap U$. We can assume that $U \cap j(S) \cong e \times [0, 1)$. Then, since the S^1 orbits collapse to points on $\text{Crit } \check{f}$, it is easy to see that $\tilde{S}^* \cap \check{f}^{-1}(U) \cong e \times D$, where $D \subset \mathbb{C}$ is the open unit disc. Thus $\tilde{S}^* \cap \check{f}^{-1}(U)$ is a 3-manifold.

In a similar way we can construct $\tilde{S} \cup f^{-1}(U)$. The orbits of the T^2 action (7) give the monodromy invariant T^2 with respect to monodromy around e (see (4)). Thus $\mathcal{F}_b/\mathcal{F}_b \cap \Lambda^*$ must coincide with such an orbit. In particular, given a section $\sigma : j(S) \cap U \rightarrow X_B$ such that $\sigma(e) \subseteq \text{Crit } f$, we can define $\tilde{S} \cap f^{-1}(U)$ to be the union of the T^2 -orbits of the points $\sigma(p)$ for all $p \in j(S) \cap U$. It can be seen that $\tilde{S} \cap f^{-1}(U) \cong e \times S^1 \times D$, and thus it is a 4-manifold.

In the following we give an explicit construction of \tilde{S} and \tilde{S}^* near an edge.

Example 7.4. Let $f : X \rightarrow \mathbb{R}^3$ be as in (6). Let $j(S) = \{x_1 = 0, x_2 \geq 0\}$ and define

$$\tilde{S} = \{z_2 = 0\}.$$

Clearly $f(\tilde{S}) = j(S)$ and it can be easily checked that the fibres of $f|_{\tilde{S}}$ are precisely the orbits of the T^2 action. Also, \tilde{S} is constructed as above by taking the section

$$\sigma(0, x_2, x_3) = (\sqrt{x_2}, 0, e^{x_3}),$$

which is well defined on $j(S)$. Similarly, let $j(S) = \{x_1 \leq 0, x_2 = 0\}$ and define

$$\tilde{S}^* = \{z_1 = \bar{z}_2, z_3 \in \mathbb{R}\}.$$

Then $f(\tilde{S}^*) = j(S)$ and the fibres of $f|_{\tilde{S}^*}$ are precisely the orbits of the S^1 action. As a section we could take

$$\check{\sigma}(x_1, 0, x_3) = (\sqrt{1 - e^{x_1}}, \sqrt{1 - e^{x_1}}, e^{x_3})$$

which is well defined on $j(S)$.

In order to be able to glue the above local models of \tilde{S} and \tilde{S}^* we should deform the given sections so that away from $j(\partial S)$ they coincide with some fixed section σ_0 , which we regard as the zero section (recall that the image of a genuine section must avoid $\text{Crit } f$). A possible choice of σ_0 could be the following

$$(34) \quad \sigma_0(x_1, x_2, x_3) = \left(\frac{\sqrt{2}(1 + e^{x_1})}{\sqrt{-x_2 + \sqrt{x_2^2 + 4(1 + e^{x_1})}}}, \frac{\sqrt{-x_2 + \sqrt{x_2^2 + 4(1 + e^{x_1})}}}{\sqrt{2}}, e^{x_3} \right).$$

Here we just show how to deform σ , but in a very similar way we could also deform $\check{\sigma}$. Let us work on the quotient of X by the T^2 -action. Let $\pi : X \rightarrow X/T^2$ be the projection. Then X/T^2 can be identified with $(\mathbb{C} - \{1\}) \times \mathbb{R}^2$, with coordinates (u, t_1, t_2) and $\pi(z_1, z_2, z_3) = (z_1 z_2, |z_1|^2 - |z_2|^2, \log |z_3|)$. Then, if we define

$$(35) \quad \bar{f}(u, t_1, t_2) = (\log |u - 1|, t_1, t_2),$$

we have $f = \bar{f} \circ \pi$. Notice that $\bar{\sigma} = \pi \circ \sigma$ and $\bar{\sigma}_0 = \pi \circ \sigma_0$, restricted to S , have the simple expressions

$$(36) \quad \bar{\sigma}(0, x_2, x_3) = (0, x_2, x_3), \quad \bar{\sigma}_0(0, x_2, x_3) = (2, x_2, x_3).$$

We can easily interpolate between these two sections by defining

$$\bar{\sigma}'(0, x_2, x_3) = (e^{i\theta(x_2)} + 1, x_2, x_3)$$

where $\theta : \mathbb{R}_{\geq 0} \rightarrow [-\pi, 0]$ is some smooth bump function which is $-\pi$ in some neighborhood of 0 and is 0 for all sufficiently large x_1 .

Here we assumed that $j(S) = \{x_1 \geq 0, x_2 = 0\}$, but in fact we could think of (x_1, x_2) as coordinates on S and let equations (36) be the definitions of the sections on $j^*(X/T^2)$.

7.3. at nodes. Let us now discuss the local models for \tilde{S} and \tilde{S}^* at a positive node of B . Consider the following example.

Example 7.5. In the conifold X_B , near a node we have local coordinates z_1, z_2, z_3, z_4 such that X_B looks like the local conifold X_0 (equation (11)) and, in the case of a positive node, the fibration $f : X_B \rightarrow B$ is defined by equation (21) with $\delta = 0$. Here $B = \mathbb{R}^3$ with coordinates (x_1, x_2, x_3) and $\Delta = \{x_1 = x_2 = 0\} \cup \{x_1 = x_3 = 0\}$. The positive node on the base is $p = (0, 0, 0)$. We assume, as required, that \tilde{S} coincides with the local collar Σ_0 defined in (17). Then, if we let

$$j(S) = \{x_1 = 0, x_2 \geq 0, x_3 \geq 0\}$$

we have $f(\tilde{S}) = j(S)$. Notice that this definition of $j(S)$ is compatible with condition (f) of Definition 7.2. Moreover, the fibres of $f|_{\tilde{S}}$ are orbits of the T^2 action (20) and these coincide with the monodromy invariant T^2 around the edges of Δ emanating from the node. It follows from condition (h) of Definition 7.2 that over $j(S - \partial S)$, \tilde{S} coincides with the lift \tilde{S}_0 defined using the section

$$\sigma(0, x_2, x_3) = (\sqrt{x_2}, 0, \sqrt{x_3}, 0).$$

Let us now construct \tilde{S}^* . Since we are working with \check{X}_B (i.e. on the quotient of the tangent bundle of B by Λ), the local model for the fibration \check{f} in a neighborhood of a positive node of B is the negative fibration on X_0 defined in Example 5.3, with $\delta = 0$. We assume that \tilde{S}^* coincides with the local collar P_0 defined in (18). Let $j(S) = \{x_1 = 0, x_2 \leq 0, x_3 \leq 0\}$. Then $f_\delta(\tilde{S}^*) = j(S)$, where f_δ is the function (26). Notice, again, that this definition of $j(S)$ is compatible with point (f) of Definition 7.2. Moreover the fibres of $f_\delta|_{\tilde{S}^*}$ are the fibres of the S^1 action (25), whose fixed points coincide with $\text{Crit } f_\delta$. It then follows that over $j(S - \partial S)$, \tilde{S}^* coincides with the lift \tilde{S}_0^* defined using the section

$$\check{\sigma}(0, x_2, x_3) = \left(\sqrt{(1 - e^{x_2})(1 - e^{x_3})}, \sqrt{(1 - e^{x_2})(1 - e^{x_3})}, 1 - e^{x_2}, 1 - e^{x_3} \right).$$

As in the case of an edge of ∂S discussed in the previous paragraph, we can perturb the sections σ and $\check{\sigma}$, defining \tilde{S} and \tilde{S}^* respectively, so that they coincide with a fixed section, away from $j(\partial S)$. We discuss only the deformation of σ . Working over the quotient with respect to the T^2 action, we have X_0/T^2 is isomorphic to $(\mathbb{C} - \{1\}) \times \mathbb{R}^2$ with projection $\pi : X_0 \rightarrow (\mathbb{C} - \{1\}) \times \mathbb{R}^2$ given by $\pi(z_1, z_2, z_3, z_4) = (z_1 z_2 + z_3 z_4, |z_1|^2 - |z_2|^2, |z_3|^2 - |z_4|^2)$. Then we have $\bar{f} = \check{f} \circ \pi$, where \bar{f} is like in (35). Now consider sections of \bar{f} , defined over $j(S)$, given by

$$(37) \quad \bar{\sigma}(0, x_2, x_3) = (0, x_2, x_3), \quad \bar{\sigma}_0(0, x_2, x_3) = (2, x_2, x_3).$$

then clearly $\bar{\sigma} = \pi \circ \sigma$, while it can be shown that $\bar{\sigma}_0$ can be lifted to a smooth section of \bar{f} (similar to (34)). Then we can interpolate $\bar{\sigma}$ and $\bar{\sigma}_0$ by defining

$$\bar{\sigma}'(0, x_2, x_3) = (e^{i\theta(x_2, x_3)} + 1, x_2, x_3)$$

where $\theta : j(S) \rightarrow [-\pi, 0]$ is some smooth bump function which is $-\pi$ in some neighborhood of ∂S and is 0 outside some slightly larger neighborhood of ∂S .

We now define \tilde{S} and \tilde{S}^* near a negative node of B . To construct \tilde{S} we work with the negative fibration f_δ is given in formula (26), with $\delta = 0$. On the other hand, to construct \tilde{S}^* we need to use the positive fibration on X_0 , i.e. the map f_δ in (21).

Example 7.6. We assume that \tilde{S} coincides with the local collar Σ_0 given in (17). Let $j(S) = \{x_1 \geq 0, x_3 = 0\}$. Then we have $f(\tilde{S}) = j(S)$. Clearly $j(S)$ satisfies condition (e) of Definition 7.2. We claim that \tilde{S} is a compactification of a lift \tilde{S}_0 . First of all notice that a lift \tilde{S}_0 can be defined. In fact the two monodromy transformations along the two legs of ∂S coincide, hence they have a common monodromy invariant T^2 . This T^2 is unique as a homology class inside a fibre of f , but in this case the fibration is not Lagrangian, so that we do not have a natural connection which allows us to parallel transport the fibres and choose a preferred representative of this class. Nevertheless we may assume that there is one connection which picks a monodromy invariant T^2 in a unique way. Now observe that f , restricted to \tilde{S} , becomes

$$f|_{\tilde{S}}(z_1, z_4) = (|z_1|^2, \log |z_3 - 1|, 0).$$

It can be seen that the fibres of this map are in the same homology class as the monodromy invariant T^2 , so they coincide with the preferred monodromy invariant T^2 up to isotopy. Therefore we can assume \tilde{S} coincides with a lift \tilde{S}_0 over points $(x_1, x_2, 0) \in j(S)$ with x_1 sufficiently big.

Let us now discuss \tilde{S}^* . As mentioned, we need to use the positive fibration (21). Let \tilde{S}^* be the local collar P_0 defined in (18). If we let $j(S) = \{x_1 \leq 0, x_2 = 0\}$, then we have $f_\delta(P_0) = j(S)$. Here the definition of $j(S)$ is compatible with point (e) of Definition 7.2. Notice moreover that

$$f_{\delta|\tilde{S}^*}(z, \bar{z}, r, s) = (\log |2rs - 1|, 0, r^2 - s^2),$$

whose fibres are S^1 orbits of the action $\xi \cdot (z_1, z_2, z_3, z_4) = (\xi z_1, \xi^{-1} z_2, z_3, z_4)$. The fixed points of this action are the components of $\text{Crit } f_\delta$ given by $L_3 = \{z_1 = z_2 = z_3 = 0\}$ and $L_4 = \{z_1 = z_2 = z_4 = 0\}$. These are mapped by f_δ onto the two edges of $j(\partial S)$ defined respectively by $\{x_1 = x_2 = 0, x_3 \leq 0\}$ and $\{x_1 = x_2 = 0, x_3 \geq 0\}$. This implies that \tilde{S}^* coincides with some lift \tilde{S}_0^* .

As in the previous examples, also in this case the sections defining \tilde{S} and \tilde{S}^* can be perturbed so that, away from $j(\partial S)$, they coincide with a fixed section.

7.4. ... at positive vertices. The models of \tilde{S} and \tilde{S}^* at a positive vertex of B are similar to previous ones. For the definition of \tilde{S} we use, as the local model for the fibration $f : X \rightarrow \mathbb{R}^3$, the positive fibration given explicitly in Example 3.4. In this case, we remarked that f is invariant with respect to the T^2 action (10) and the T^2 orbits coincide with the monodromy invariant T^2 . For the definition of \tilde{S}^* inside \check{X}_B we need to use the negative fibration of Example 3.3 (see Remark 3.6).

Let us describe the local models for \tilde{S} and \tilde{S}^* explicitly for this case.

Example 7.7. Let $j(S) = \{x_1 = 0, x_2 \leq 0, x_3 \leq 0\}$, then we have that $j(\partial S)$ coincides with two of the legs of Δ . Now let

$$\tilde{S} = \{z_1 = 0\}.$$

Then, again, one can see that the fibres of $f|_{\tilde{S}}$ are precisely orbits of the T^2 action. This shows that \tilde{S} coincides with a lift \tilde{S}_0 away from $j(\partial S)$.

Let us now describe \tilde{S}^* . We consider the topological description of the negative fibration given at the beginning of Example 3.3. Recall that we have $\bar{Y} = T^2 \times \mathbb{R}^2$ and that we have the “pair of pants” $\Sigma \subset \bar{Y}$ which is mapped onto Δ by the projection onto \mathbb{R}^2 . Let (x_2, x_3) be the coordinates on \mathbb{R}^2 . Then we consider $Y = \bar{Y} \times \mathbb{R}$ and we denote by x_1 the extra \mathbb{R} -coordinate. We identify \bar{Y} with $\{x_1 = 0\}$. We then consider the space X with the S^1 action, whose quotient is Y and such that the fixed point set coincides with Σ . Now let $j(S) = \{x_1 = 0, x_2 \leq 0, x_3 \leq 0\}$. Then $\partial S \subset \Delta$ and this definition of $j(S)$ is compatible with Definition 7.2. Now consider a section $\tilde{\sigma} : j(S) \rightarrow Y$ such that $\tilde{\sigma}(j(\partial S)) \subset \Sigma$. We define $\tilde{S}^* = \pi^{-1}(\tilde{\sigma}(j(\partial S)))$, where $\pi : X \rightarrow Y$ is the quotient by the S^1 action. It can be easily seen that in this case \tilde{S}^* is homeomorphic to $\mathbb{C} \times \mathbb{R}$, since the orbits of the S^1 action collapse to points over Σ .

Again, \tilde{S} and \tilde{S}^* can be perturbed so that, away from $j(\partial S)$, they coincide with the lifts \tilde{S}_0 and \tilde{S}_0^* constructed from a fixed section. Moreover, one can assume that up to local isotopy, near a positive vertex, $j(S)$ is like in the above example.

7.5. ... at negative vertices. We now describe models for \tilde{S} and \tilde{S}^* near a boundary vertex $p \in S$ (Definition 7.1, condition (e)). We must have (point (d) of Definition 7.2) that $j(p)$ is a negative vertex of B . Then, to construct \tilde{S} inside X_B , we use as the local model for the fibration, the negative fibration of Example 3.3. To construct \tilde{S}^* inside \check{X}_B we use the positive fibration of Example 3.4.

Let us start describing \tilde{S} . Let us consider the topological description of the negative fibration $f : X \rightarrow \mathbb{R}^3$ given at the beginning of Example 3.3. Then let

$$j(S) = \Delta \times \mathbb{R}_{\geq 0} \subseteq \mathbb{R}^3.$$

Since Δ is homeomorphic to a tropical line V_1 , we have that $(0, 0, 0) \in j(S)$ is the image of a boundary vertex (here we assume that $(0, 0)$ is the vertex of Δ). Now recall that $\Sigma \subset \bar{Y}$ is the “pair of pants” which is mapped by P to Δ and that $P^{-1}(\Delta_i)$ is the cylinder $S^1 \times \Delta_i$, where $S^1 \subset T^2$ represents the class $-e_3$, $-e_2$ and $e_2 + e_3$ when $i = 1, 2, 3$ respectively. If $\pi : X \rightarrow Y$ is the S^1 quotient defining the space X , then we define the lift $\tilde{S} \subset X$ of $j(S)$ to be

$$\tilde{S} = \pi^{-1}(\Sigma \times \mathbb{R}_{\geq 0}).$$

We have that \tilde{S} is homeomorphic to $\Sigma \times D$, where D is a disc in \mathbb{R}^2 . Now consider $f|_{\tilde{S}} : \tilde{S} \rightarrow j(S)$ and let $q \in \Delta_1 \times \mathbb{R}_{>0} \subset j(S)$. Notice that $f|_{\tilde{S}}^{-1}(q) = S^1 \times S^1$, where the first factor is a circle representing $-e_3$ and the second factor is a circle representing e_1 , the class of an orbit of the S^1 -action. These two cycles are precisely the monodromy invariant ones with respect to monodromy around Δ_1 . Similarly we can argue about $f|_{\tilde{S}}^{-1}(q) = S^1 \times S^1$, with $q \in \Delta_j \times \mathbb{R}_{>0}$ and $j = 2, 3$. Thus $f|_{\tilde{S}}^{-1}(\Delta_i \times \mathbb{R}_{\geq 0})$ coincides with the construction of \tilde{S} done in the case of the smooth boundary points in Δ_i . Notice moreover that for all $q \in \Delta_i \times \mathbb{R}_{>0}$, which are sufficiently away from $\Delta_i \times \{0\}$, we can assume that $f|_{\tilde{S}}^{-1}(q) \cong T^2$ coincides, up to translation by a section, with the unique linear subspace of $f^{-1}(q)$ which is monodromy invariant with respect to a flat connection on the space of smooth fibres. We can also perturb \tilde{S} , so that away from $j(\partial S)$ it coincides with a lift \tilde{S}_0 constructed from a fixed section. Observe that the points on $(0, 0) \times \mathbb{R}_{>0} \subset j(S)$ are images of interior edge points of S . A more detailed description of \tilde{S} over such points will be given in §7.7.

Let us now construct \tilde{S}^* , where we use the positive fibration $f : X \rightarrow \mathbb{R}^3$ as the model fibration. In this case, f is T^2 invariant with respect to the action (10). We let $j(S) = \mathbb{R}_{\geq 0} \times \Delta$ and let $\tilde{\sigma} : j(S) \rightarrow X$ be a section such that $j(\partial S) \subset \text{Crit } f$. Notice that $j(\partial S) = \Delta$ and $\text{Crit } f$ consists of the union of the sets $L_j = \{z_i = 0, i \neq j\}$, with $j = 1, 2, 3$. Each of these is mapped to one of the legs of Δ . Denote by Δ_j the leg of Δ which is the image of L_j . For each $j = 1, 2, 3$, there is a unique subgroup of the torus T^2 which leaves all points of L_j fixed. For instance, the subgroup which leaves the points of L_3 fixed is $\{\xi_1 \xi_2 = 1\}$. Let us denote by $G_j \cong S^1$ such a subgroup. Now define

$$\tilde{S}_j^* = \bigcup_{p \in \mathbb{R}_{\geq 0} \times \Delta_j} G_j \cdot \tilde{\sigma}(p).$$

It can be checked that, since the orbits of G_j collapse to single points on L_j , we have $\tilde{S}_j^* \cong D \times \mathbb{R}_{\geq 0}$. Where $D \times \{0\}$ corresponds to the union of the G_j -orbits of points $\tilde{\sigma}(p)$ for all $p \in \mathbb{R}_{\geq 0} \times \{(0, 0)\}$. Observe that there is a suitable choice of orientations of the G_j 's such that, in $H_1(T^2, \mathbb{Z})$, the classes $[G_j]$ satisfy $[G_1] + [G_2] + [G_3] = 0$. Let $Q \subset T^2$ be a 2-chain such that $\partial Q = [G_1] + [G_2] + [G_3]$ and consider the following closed subset of X :

$$C = \bigcup_{p \in \mathbb{R}_{\geq 0} \times \{(0, 0)\}} Q \cdot \tilde{\sigma}(p)$$

Define

$$\tilde{S}^* = C \cup \tilde{S}_1^* \cup \tilde{S}_2^* \cup \tilde{S}_3^*.$$

Observe that Q can be chosen so that \tilde{S}^* is a submanifold homeomorphic to \mathbb{R}^3 . This can be seen as follows. One can choose Q homeomorphic to a standard 2-simplex with the vertices identified (see Figure 15 (a)). Then, since the T^2 orbit of $\tilde{\sigma}(0, 0, 0)$ collapses to a point, we have that C is homeomorphic to the cone of Q , i.e. to $\mathbb{R}_{\geq 0} \times Q$ after the subset $\{0\} \times Q$ is collapsed to a single

point. One can then see that after attaching to this space the sets \tilde{S}_j^* we obtain something homeomorphic to \mathbb{R}^3 .

7.6. Pair of pants. Before engaging in the proof of Theorem 7.3 we generalize the latter argument and give a slightly more general “pair of pants” construction. Let $v_1, v_2, v_3 \in \mathbb{R}^2$ be primitive integral vectors, spanning \mathbb{R}^2 , such that

$$(38) \quad v_1 + v_2 + v_3 = 0.$$

Given $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, for each $j = 1, 2, 3$ the subspace $\mathbb{R}v_j$ covers a subgroup $G_j \cong S^1 \subseteq T^2$, with orientation induced by v_j . Notice that since all v_j ’s are primitive, under the canonical isomorphism $\mathbb{Z}^2 \cong H_1(T^2, \mathbb{Z})$, the subgroups G_j represent the class v_j . Let $V^1 \subseteq \mathbb{R}^2$ be the tropical line as defined at the beginning of §7.1 and denote by $p = (0, 0)$ its vertex and by D_1, D_2 and D_3 the three edges of V^1 emanating from p , indexed in anticlockwise order. Let $X = \mathbb{R}^2 \times T^2$ and consider the following subset of X :

$$\Sigma_0 = \bigcup_{j=1}^3 D_j \times G_j.$$

We have the following

Lemma 7.8. There exists $Q \subset \{p\} \times T^2$ such that

$$\Sigma = \Sigma_0 \cup Q$$

is an embedded piecewise smooth submanifold of X .

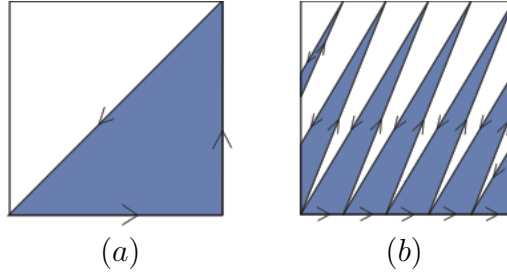


FIGURE 15. The region Q (in blue): (a) when $\{v_1, v_2\}$ is a \mathbb{Z}^2 -basis, (b) when it is not.

Proof. Notice that (38) implies that there exists a chain complex Q such that $\partial Q = \cup_j G_j$. We want to show that Q can be chosen so that Σ is a manifold. If $\{v_1, v_2\}$ forms a \mathbb{Z}^2 -basis, choose Q to be the triangle pictured in Figure 15 (a). Then Σ is a piecewise smooth submanifold of X . Notice that in principle one could also choose a different triangle, i.e. the other half of the square, but in this case its orientation should be opposite to the one induced by the choice of ordering of the v_j ’s. In particular the ordering allows a canonical choice of Q .

Let us now discuss the case where $\{v_1, v_2\}$ does not form a \mathbb{Z}^2 -basis. In Figure 15 we have pictured the choice of Q for $v_1 = (1, 0)$ and $v_2 = (2, 5)$.

Notice that in this case $v_3 = (-3, -5)$ is also primitive, as required in the hypothesis. Suppose that v_1 and v_2 generate a sublattice of \mathbb{Z}^2 of index m . Let N be the lattice generated by $\hat{v}_1 = m^{-1}v_1$ and $\hat{v}_2 = m^{-1}v_2$ and let $T' = \mathbb{R}^2/N$. Since $\mathbb{Z}^2 \subset N$, the identity of \mathbb{R}^2 induces a covering $\pi : T^2 \rightarrow T'$ of degree m . Let $\hat{v}_3 = -\hat{v}_1 - \hat{v}_2$ and let $G'_j \subseteq T'$ be the group covered by $\mathbb{R}\hat{v}_j$. Then, since $\{\hat{v}_1, \hat{v}_2\}$ forms a basis of N , we can choose the triangle Q' such that $\partial Q' = \cup_j G'_j$ as above. Now let $Q = \pi^{-1}(Q')$. It can be checked that $\partial Q = \cup_j G_j$. Moreover Σ , defined with this choice of Q , is a piecewise smooth submanifold of X . \square

7.7. Proof of Theorem 7.3. In the previous discussion we have shown how the zero sections of f and \check{f} can be perturbed in a neighborhood of $j(\partial S)$ so that \tilde{S} and \tilde{S}^* can be constructed over points on $j(\partial S)$. Now we need to show how \tilde{S} and \tilde{S}^* can be defined over the remaining points of $j(S)$. We have the following cases: p is one of the smooth points q_1, \dots, q_r in condition (b) of Definition 7.2; p is an interior vertex of S ; p is an interior edge point of S . In all these cases we will define a suitable closed subset Q_p (resp. Q_p^*) of the fibre $f^{-1}(j(p))$ (resp. $\check{f}^{-1}(j(p))$) such that

$$\tilde{S}_0 \cup (\cup_p Q_p) \quad (\text{resp. } \tilde{S}_0^* \cup (\cup_p Q_p^*))$$

is a submanifold. Let us take an interior edge $e \subset S$ and a point $p \in e$. We apply point (g) of Definition 7.2. Given some orientation on e and a small connected neighborhood U of p in S , $j(U \cap S_{sm})$ has three connected components indexed with a unique cyclic order. Moreover, we have the vector fields v_1, v_2 and v_3 satisfying the balancing equation. We may assume, without loss of generality, that all ϵ_k 's in the balancing equation are 1.

Let us first define Q_p^* . Inside $\check{f}^{-1}(j(p)) = T_{j(p)}B/\Lambda_{j(p)}$, the span of the vectors v_1, v_2 generates a two torus T and we may consider its subgroups G_k , $k = 1, 2, 3$, generated by the v_k 's. Notice that if we parallel transport G_k to a point q of the k -th component of $j(U \cap S_{sm})$, then $G_k = \mathcal{V}_q/\mathcal{V}_q \cap \Lambda_q$. The balancing condition is equivalent to saying that in $H_1(T, \mathbb{Z}^2)$ we have $[G_1] + [G_2] + [G_3] = 0$. Then we may define $Q_p^* \subset T$ as in the proof of Lemma 7.8. Notice that the choice of Q_p^* is canonical, given by the ordering of the v_k 's. Notice also that it is independent of the choice of orientation of the edge e . In fact, changing the orientation of e inverts the cyclic ordering but also changes all the signs ϵ_k appearing in the balancing condition and hence the orientations of the G_k 's. It is easy to see that attaching Q_p^* to \tilde{S}_0^* for all interior edge points p , produces a submanifold. This is how we define \tilde{S}^* over interior edges.

Now define Q_p . Inside $f^{-1}(j(p))$ we have a triple of 2-tori T_1, T_2 and T_3 defined by v_1, v_2 and v_3 via the bundle \mathcal{F} . Recall that both fibres $\check{f}^{-1}(j(p))$ and $f^{-1}(j(p))$ are canonically oriented. Let $w \in T_p^*B$ be primitive and integral, such that $w(v_1) = w(v_2) = 0$. The sign of w can be chosen by imposing that $\langle w, v \rangle > 0$ for all $v \in T_{j(p)}B$ such that $\{v_1, v_2, v\}$ forms an oriented \mathbb{R} -basis of $T_{j(p)}B$. Notice that $\mathbb{R}w/\mathbb{Z}w = T_1 \cap T_2 \cap T_3$. The vector v_k and the orientation on $f^{-1}(j(p))$ induce an orientation on T_k . Now let T be the two torus obtained as the quotient of $f^{-1}(j(p))$ by $\mathbb{R}w/\mathbb{Z}w$. Then T_k is mapped

to a subgroup $G_k \subset T$, with an orientation. Again, the balancing condition implies $[G_1] + [G_2] + [G_3] = 0$ in $H_1(T, \mathbb{Z}^2)$. Then, as in the proof of Lemma 7.8, we can construct Q'_p such that $\partial Q'_p = \cup_k G_k$. Define $Q_p \subset f^{-1}(j(p))$ to be the pre-image of Q'_p via the quotient map $f^{-1}(j(p)) \rightarrow T$. Then clearly $\partial Q_p = \cup_k T_k$. It is not difficult to show that attaching Q_p to \tilde{S}_0 for all interior edge points p we obtain a submanifold. This is how we define \tilde{S} over interior edges.

In case the interior edge ends on a boundary vertex, we should show that these constructions coincide with the ones already given in a neighborhood of a boundary vertex (see §7.5). In the case of \tilde{S}^* , the construction just given and the construction given in §7.5 coincide exactly. In fact, by parallel transport, the vector fields v_1 and v_2 can be defined on a neighborhood U of the edge e and they generate a T^2 action on $\tilde{f}^{-1}(U)$. The torus T mentioned above in the definition of Q_p^* is an orbit of this T^2 action. If e ends on a boundary vertex it can be seen that this T^2 action coincides with the one used to construct \tilde{S}^* in §7.5. Moreover, also the subgroups G_k , defined above, coincide with subgroups G_k in §7.5. Therefore the two constructions coincide.

In the case of \tilde{S} , the two constructions do not strictly coincide, but they are isotopic. Therefore they can be made to coincide by using this isotopy along the edge e . First of all observe that the three tori T_1, T_2 and T_3 near a boundary vertex must coincide with the monodromy invariant ones around each leg of Δ . In §7.5 we had $j(S) = \Delta \times \mathbb{R}_{\geq 0}$ and interior edge points of $j(S)$ are of type $q = (0, 0, t)$ where $(0, 0)$ is the vertex of Δ and $t > 0$. Also we had $\tilde{S} = \pi^{-1}(\Sigma \times \mathbb{R}_{\geq 0})$, where Σ is the “pair of pants” constructed in Example 3.3. The main difference between the construction of \tilde{S} given above and the one given in §7.5 is precisely in the way we defined Σ . In fact, in Example 3.3 the legs of Σ are glued together so that, if b_0 is the vertex of Δ , then $\Sigma \cap (T^2 \times \{b_0\})$ is a “figure eight”. We could make a different choice, e.g. we could assume that the three circles, forming the legs of Σ , come together at $T^2 \times \{b_0\}$ as linear subspaces. Then we could assume that $\Sigma \cap (T^2 \times \{b_0\})$ is some closed set Q' , such as a triangle or a pair of triangles. The main point is that any such choice would be equivalent up to isotopy and one can choose Q' so that we obtain the same construction of \tilde{S} as described above. Notice also that the S^1 action coincides with the action of $\mathbb{R}w/\mathbb{Z}w$. In particular one can use the isotopy to interpolate the two constructions as we move away from a boundary vertex.

Now let $p \in S$ be an interior vertex. Let us define Q_p^* first. There are four interior edges of S meeting at p . Moreover, given small connected neighborhood U of p in S , $j(U \cap S_{sm})$ has 6 connected components. Let us orient each interior edge emanating from p in the direction moving away from p . Let v_1, \dots, v_6 denote the vector field v restricted to the six components of $j(U \cap S_{sm})$. We can assume that the indices have been chosen in such a way that the following ordered sets $\{v_1, v_2, v_3\}$, $\{v_3, v_4, v_5\}$, $\{v_1, v_5, v_6\}$ and $\{v_2, v_6, v_4\}$ correspond to the triples meeting at each one of the four interior edges, ordered according to the cyclic ordering imposed by their orientations. Without loss of generality,

one can see that the sign rule in the balancing condition gives the following equations

$$v_1 + v_2 + v_3 = 0, \quad -v_3 - v_5 - v_4 = 0, \quad -v_1 + v_6 + v_5 = 0, \quad -v_2 + v_4 - v_6 = 0.$$

Now let Q_1^*, Q_2^*, Q_3^* and Q_4^* denote the four subsets of $\check{f}^{-1}(j(p))$ constructed as above from these triples of vectors. These are obviously the limits of the sets $Q_{p'}^*$ constructed along the four edges as p' approaches the vertex p . Using point (j) of Definition 7.2 we can assume that v_1, v_2 and v_4 are linearly independent. If v_1, v_2 and v_4 are a \mathbb{Z} -basis for Λ_p then it can be calculated that $\cup_k Q_k^*$ is the boundary of a 3-simplex immersed in $\check{f}^{-1}(j(p))$. In this case we define Q_p^* to be this 3-simplex. More generally one can use an argument similar to the one used in the proof of Lemma 7.8 to find a suitable Q_p^* such that $\partial Q_p^* = \cup_k Q_k^*$. One can show that attaching to \tilde{S}_0^* this Q_p^* and the sets $Q_{p'}^*$ as above for all interior edge points, we obtain a submanifold.

Now let us define Q_p . For each point on the four edges emanating from p , the previous construction gave a 3-chain whose boundary is the union of the three 2-tori corresponding to that edge. These four 3-chains come together at the point p as subsets of $f^{-1}(j(p))$. Denote them by Q_j , $j = 1, \dots, 4$ and define $Q_p = \cup_{j=1}^4 Q_j$. It can be verified that attaching to \tilde{S}_0 this Q_p and the sets $Q_{p'}$ as above for all interior edge points p' , we obtain a submanifold.

It remains to define Q_p and Q_p^* when p is one of the points $\{q_1, \dots, q_r\}$ of point (b) of Definition 7.2. Let us define Q_p first. Given the condition on the monodromy of \mathcal{F} around p (condition (i) of Definition 7.2) it is natural to guess that Q_p must be an I_1 fibre. This can be shown as follows. Let us use the construction in Example 3.2 of the generic-singular fibration over an edge of Δ . Then the base of the fibration is $U = D \times (0, 1)$ and the quotient of X by the S^1 action is $Y = U \times T^2$. Let $e_2, e_3 \in H_1(T^2, \mathbb{Z})$ and Σ be defined as in Example 3.2. If e_1 is the orbit of the S^1 action, then monodromy of f is given by (4) in the basis e_1, e_2, e_3 of $H_1(X_b, \mathbb{Z})$. Now let $\{h_1, h_2\}$ be an integral basis of \mathcal{F} with respect to which monodromy of \mathcal{F} around p has the given form. Then it can be shown that h_2 represents the class e_1 and h_1 represents the class $e_2 + ae_3$ for some $a \in \mathbb{Z}$. Then by change of basis of $H_1(T^2, \mathbb{Z})$, w.l.o.g. we may as well assume that h_1 represents e_2 . We may also assume that $j(S) \cap U = D \times \{1/2\}$. If we consider a circle $S^1 \subseteq T^2$ representing the class e_2 , then we may define $\tilde{S} = \pi^{-1}(D \times \{1/2\} \times S^1)$, where $\pi : X \rightarrow Y$ is the projection with respect to the S^1 action. Then \tilde{S} has an I_1 fibre over $p = (0, 1/2) \in D \times (0, 1) = U$ and it is a submanifold.

To define Q_p^* , we use the same model for the generic singular fibration. but since we are working on the tangent bundle one can see that condition (i) of Definition 7.2 implies that the vector field v corresponds to the class e_3 . In particular v is monodromy invariant. This implies that v induces an S^1 action on $\check{f}^{-1}(U)$ where U is a neighborhood of $j(p)$. Clearly, away from p , the circles $\mathcal{V}_q/\mathcal{V}_q \cap \Lambda_q$ defining \tilde{S}_0^* are orbits of this S^1 action. We define Q_p^* to be the orbit

of $\sigma_0(j(p))$ with respect to this S^1 action, where σ_0 is the zero section. Clearly attaching Q_p^* to \tilde{S}_0^* gives a submanifold.

We have now explained how to define \tilde{S} and \tilde{S}^* over all points of S . We complete the argument with a last remark on how to make the constructions coincide on the overlaps between the various open sets where \tilde{S} or \tilde{S}^* were constructed. For instance in the case of an edge of ∂S ending on either a node, positive or negative fibre, we have to make sure that the constructions coincide. It is clear though that in all such cases the sections used can be interpolated with the aid of usual bump functions. Also, in the cases where the choice of the T^2 or S^1 fibres are not canonically given as an orbit of an action (e.g. in the case of a negative node or negative vertex), then we can interpolate choices via some isotopy along the edge. This concludes the proof.

Remark 7.9. We point out that our construction of \tilde{S} and the fibration $f_{|\tilde{S}} : \tilde{S} \rightarrow S$ is very similar to the fibration constructed by G. Mikhalkin in [23] of a hypersurface in $(\mathbb{C}^*)^n$ over its “non-archimedean” amoeba.

8. SIMULTANEOUS RESOLUTIONS/SMOOTHINGS OF NODES

We discuss here the problem of simultaneously resolving a set of nodes in a tropical conifold (B, \mathcal{P}, ϕ) . More precisely, given a set of nodes in B and hence in X_B , the topological conifold associated to B , we want to construct a new tropical manifold $(B', \mathcal{P}', \phi')$ such that the topological compactification $X_{B'}$ is homomorphic to the resolution of the corresponding nodes in X_B . To achieve this we need to change the affine structure on B so that a neighborhood of the node is replaced by a neighborhood which is (locally) affine isomorphic to its tropical resolution described in §6.3 and §6.4. This is not a local problem. In fact we should expect that in the compact case, this problem has global obstructions. Intuitively, resolving a node implies the insertion of an extra 2-dimensional face (in the positive case) or an extra 3-dimensional polytope (in the negative case), causing a modification of nearby polytopes which propagates away from the node.

As we have discussed in §6.3 and §6.4, the local resolution of a node can be achieved by smoothing the mirror node and then applying discrete Legendre transform. Therefore, the problem of tropically resolving a set of nodes is equivalent to the one of tropically smoothing the mirror ones. In this sense the phrase “simultaneously resolving/smoothing a set of nodes” also means simultaneously resolving the nodes on B *and* smoothing the mirror nodes on \check{B} . In Theorem 7.3 we have proved that the existence of tropical 2-cycles in B containing the nodes guarantees that the topological obstructions to resolve the nodes in X_B and to smooth the mirror ones in $X_{\check{B}}$ vanish simultaneously. This suggests the idea that the existence of tropical 2-cycles could imply the vanishing of the obstructions to the tropical resolution of nodes.

So, let p_1, \dots, p_{k+s} be a set of nodes of \check{B} , where p_1, \dots, p_k are negative and p_{k+1}, \dots, p_{k+s} are positive. The negative ones are the barycenters of square 2-dimensional faces e_1, \dots, e_k and the positive ones are barycenters of 1-dimensional

edges $\ell_{k+1}, \dots, \ell_{k+s}$. Then, to simultaneously smooth these nodes we want to do the following:

- a) find a refinement $\check{\mathcal{P}}'$ of $\check{\mathcal{P}}$ inducing a diagonal subdivision of e_1, \dots, e_k and the barycentric subdivision of $\ell_{k+1}, \dots, \ell_{k+s}$;
- b) define a suitable fan structure, compatible with $\check{\mathcal{P}}'$, at the barycenters of the edges $\ell_{k+1}, \dots, \ell_{k+s}$ which has the effect of changing the discriminant locus as in §6.4;
- c) construct an MPL-function $\check{\phi}'$, strictly convex with respect to $\check{\mathcal{P}}'$.

This has to be done so that we obtain a new tropical manifold (or conifold, if there are other nodes) $(\check{B}', \check{\mathcal{P}}', \check{\phi}')$ and the corresponding manifold (or conifold) $X_{\check{B}'}$ is homeomorphic to the smoothing of $X_{\check{B}}$ at the nodes corresponding to p_1, \dots, p_{k+s} . If this can be done, then the mirror $(B', \mathcal{P}', \phi')$ of $(\check{B}', \check{\mathcal{P}}', \check{\phi}')$ gives the simultaneous resolution of the mirror nodes. We remark that in point (a) the subdivision of the edges $\ell_{k+1}, \dots, \ell_{k+s}$ does not have to be necessarily barycentric, it may simply be a subdivision given by adding one vertex at an interior integral point. During the process, in order to preserve integrality, it may be necessary to rescale all polytopes by some common factor

Definition 8.1. A given set of nodes p_1, \dots, p_k of a tropical conifold (B, \mathcal{P}, ϕ) is said to be *related* if there exist tropical 2-cycles $(S_1, j_1) \dots, (S_r, j_r)$ such that $\cup_{m=1}^r j_m(S_m)$ contains the nodes p_1, \dots, p_k and no other nodes.

In view of this definition and Theorem 7.3 we can conjecture the following

Conjecture 8.2. A set of nodes p_1, \dots, p_k in a tropical conifold (B, \mathcal{P}, ϕ) can be simultaneously resolved by the above process if and only if they are related.

Although we believe this conjecture to be morally true, we do not exclude that some refinement in our definition of tropical 2-cycle will be necessary. In the following we prove this conjecture for certain configurations of nodes which are related.

8.1. Some special cases. Suppose that there is a polytope $P \in \mathcal{P}$ of the type $P = L \times [0, m]$, where L is a 2-dimensional convex integral polytope in \mathbb{R}^2 and consider $S = L \times \{m/2\}$. We assume that $\partial S \subset \Delta$ and that the vertices p_1, \dots, p_k of S are positive nodes. These nodes are related (see Figure 16) by the tropical 2-cycle (S, j, v) , where j is the inclusion and v is the vector field parallel to the edges containing the nodes.

Theorem 8.3. If positive nodes p_1, \dots, p_k in a tropical conifold (B, \mathcal{P}, ϕ) are in a configuration as above, then they can be tropically resolved.

Proof. We smooth the mirror nodes. First observe that since the positive nodes coincide with the vertices of S and $\partial S \subseteq \Delta$, then L (and therefore S) must be a Delzant polytope. This means that the tangent wedge at each vertex of L is generated by primitive integral vectors which form a \mathbb{Z}^2 basis. Let q_1, \dots, q_k be the vertices of L , ordered so that q_j and q_{j+1} belong to the same edge (and indices are cyclic). Then the edges of P containing the nodes are given by

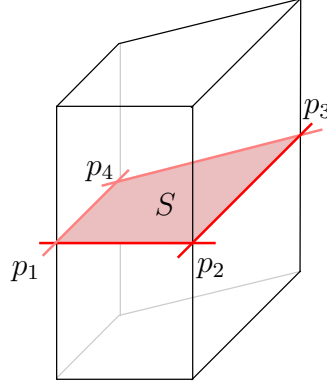


FIGURE 16.

$e_j = q_j \times [0, m]$ and we denote the vertices of these edges by $q_j^+ = (q_j, m)$ and $q_j^- = (q_j, 0)$. Let us denote by Q_j^+ and Q_j^- the 3-dimensional polytopes of $\check{\mathcal{P}}$ which are dual to the vertices q_j^+ and q_j^- respectively. It is clear that Q_j^+ and Q_j^- will have as common face the square face \check{e}_j which is dual to the edge e_j . Now all the faces \check{e}_j intersect in a common vertex v_0 which is dual to the polytope P . Moreover, \check{e}_{j-1} and \check{e}_j intersect in an edge, which we denote ℓ_j . One of the vertices of ℓ_j is v_0 and we denote the other one by v_j (see Figure 17). The fan structure at v_0 is given by the normal fan of P , which we denote Σ_P . If $\Sigma_L \subset \mathbb{R}^2$ denotes the normal fan of L , with 2-dimensional cones C_1, \dots, C_k , then Σ_P is clearly given by the 3-dimensional cones $C_j^+ = C_j \times [0, +\infty)$ and $C_j^- = C_j \times (-\infty, 0]$. We have that C_j^+ and C_j^- are the tangent wedges at v_0 of the polytopes Q_j^+ and Q_j^- respectively. Notice that emanating from v_0 we have the edges ℓ_j , all lying in a plane, plus two more edges transversal to this plane which we denote ℓ^+ and ℓ^- , belonging to Q_j^+ and Q_j^- respectively. In affine coordinates ℓ^+ and ℓ^- have opposite directions.

We now describe the refinement $\check{\mathcal{P}}'$ of $\check{\mathcal{P}}$. We subdivide the faces \check{e}_j by taking the diagonal joining v_j to v_{j+1} . We extend this subdivision as follows. Let us denote by v^+ and v^- the interior integral points of the edges ℓ^+ and ℓ^- respectively, closest to v_0 . Now subdivide each Q_j^\pm in two convex polytopes: one being $V_j^\pm = \text{Conv}(v^\pm, v_0, v_j, v_{j+1})$ and the other the closure of its complement. It is clear that this gives a well defined refinement $\check{\mathcal{P}}'$ of $\check{\mathcal{P}}$ which extends the given diagonal subdivision of the faces. Notice that the union of all the diagonals from v_j to v_{j+1} encloses a 2-dimensional region containing the vertex v_0 . We denote this region by \check{S} . We can assume, perhaps after rescaling \check{B} , that the only parts of the discriminant locus $\check{\Delta}$ intersected by V_j^+ and V_j^- are those contained in \check{S} . This ensures that the decomposition $\check{\mathcal{P}}'$ alters the discriminant locus only at the nodes contained in the faces \check{e}_j .

We now explain the construction of $\check{\phi}'$. Let us sketch the main ideas. First we construct a multivalued piecewise linear function $\check{\phi}$, which we may regard as “supported in a neighborhood of \check{S} ”. It can be described as follows. Label the

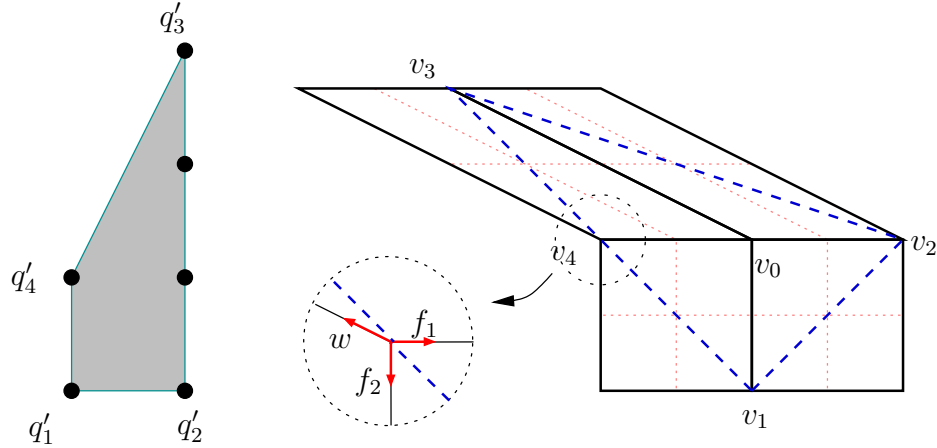


FIGURE 17. On the left we see the polytope L and on the right the square faces $\check{e}_1, \dots, \check{e}_4$ in the mirror. The dashed diagonals give the subdivision and we have also drawn the discriminant locus (in red).

edges emanating from v_0 by a pair of integers, e.g. (n_j, m_j) for the edges ℓ_j , $j = 1, \dots, k$ and (n^\pm, m^\pm) for the edges ℓ^\pm . Then, on the fan at v_0 , $\tilde{\phi}$ maps the integral generator of the one dimensional cone corresponding to ℓ_j (resp. ℓ^\pm) to m_j (resp. m^\pm). On the fan at v_j , $\tilde{\phi}$ maps the generator of the one dimensional cone corresponding to ℓ_j to n_j and all other edges to zero. On the fan at v^\pm , $\tilde{\phi}$ maps the generator of the one dimensional cone corresponding to ℓ^\pm to n^\pm and all other edges to zero. At all remaining vertices of $\check{\mathcal{P}}'$, $\tilde{\phi}$ is defined to be zero. Clearly not all choices of labels of the edges give a well defined $\tilde{\phi}$, but we claim that if n is the integral length of the edges ℓ_j (independent of j) and we choose $(n_j, m_j) = (1, -1)$, $(n^+, m^+) = (n, -2n + 1)$ and $(n^-, m^-) = (n, 0)$ then $\tilde{\phi}$ is a well defined multivalued piecewise linear function. Once we have proved this, we define

$$(39) \quad \check{\phi}' = N\check{\phi} + \tilde{\phi},$$

for some integer N and we claim that for N sufficiently big, $\check{\phi}'$ is strictly convex.

So let us prove that $\tilde{\phi}$, as described above, is well defined. First we have to be careful that on all fans, the given integers define, on each 3-dimensional cone, an integral linear map. Next we need to show that, along every edge of $\check{\mathcal{P}}'$ connecting vertices v and v' , the quotient functions with respect to that edge, computed at v and v' , match. At a vertex v_j , construct a basis $\{f_1, f_2, f_3\}$ for $T_{v_j}\check{B}$ as follows. Let f_1 be the integral generator of the edge ℓ_j and f_2 the integral generator of the other edge of \check{e}_j adjacent to v_j (hence parallel to ℓ_{j+1}). Notice that if we denote by w the integral generator of the other edge of \check{e}_{j-1} adjacent to v_j (hence parallel to ℓ_{j-1}) then $w = -f_2 + k_j f_1$ for some k_j (see Figure 17). Now at v_0 , consider the vector $v^+ - v_0$ and parallel

transport it to $T_{v_j}\tilde{B}$ via a curve from v_0 to v_j passing into V_j^+ . We define f_3 to be the resulting vector. Notice that, if we parallel transport f_3 back along the same curve and then again to v_j along a curve passing into V_j^- , then we obtain $f_3 + f_1$. This is the effect of monodromy around the component of the discriminant locus passing through the edge ℓ_j . This implies that if we parallel transport $v^- - v_0$ to v_j along a curve passing into V_j^- then we obtain $-f_3 - f_1$, since in affine coordinates at v_0 we have $v^- - v_0 = -(v^+ - v_0)$. Notice that the tangent direction to the edge from v_j to v^+ at v_j is $nf_1 + f_3$, where n is the integral length of ℓ_j . Similarly the tangent direction to the edge from v_j to v^- is $(n-1)f_1 - f_3$. Finally, in the fan structure at v_j , the tangent wedges of the polytopes V_j^+ , V_j^- , V_{j-1}^- , V_{j-1}^+ correspond respectively to the cones

$$\begin{aligned} &\text{Cone}(f_1, f_1 + f_2, nf_1 + f_3), \quad \text{Cone}(f_1, f_1 + f_2, (n-1)f_1 - f_3), \\ &\text{Cone}(f_1, w + f_1, (n-1)f_1 - f_3), \quad \text{Cone}(f_1, w + f_1, nf_1 + f_3). \end{aligned}$$

As defined, $\tilde{\phi}$ has value 1 at f_1 and zero at all other one dimensional cones of the fan. Clearly, this uniquely defines integral linear maps on each cone.

The quotient fan along the edge from v_j to v_0 , i.e. along f_1 , can be seen to be given by the four cones $\text{Cone}([f_2], [f_3])$, $\text{Cone}([f_2], -[f_3])$, $\text{Cone}(-[f_2], -[f_3])$ and $\text{Cone}(-[f_2], [f_3])$, where $[\cdot]$ denotes the class in $\mathbb{R}^3/\mathbb{R}f_1$. A calculation shows that the quotient of $\tilde{\phi}$ maps $[f_2]$ and $[f_3]$ to zero, $-[f_3]$ to $-2n+1$ and $-[f_2]$ to $-k_j-2$.

We now compute the quotient along the edge from v_j to v^+ , i.e. along $nf_1 + f_3$. We choose a basis for the quotient space to be $\{[f_1], [f_2]\}$. The quotient fan will have four maximal cones, since the edge from v_j to v^+ is contained in four maximal polytopes, two of which are V_{j-1}^+ and V_j^+ . Since $[f_1 + f_2] = [f_1] + [f_2]$ and $[w + f_1] = (k_j + 1)[f_1] - [f_2]$, the four cones are $\text{Cone}([f_1], [f_1] + [f_2])$, $\text{Cone}([f_1], (k_j + 1)[f_1] - [f_2])$, $\text{Cone}(-[f_1], [f_1] + [f_2])$ and $\text{Cone}(-[f_1], (k_j + 1)[f_1] - [f_2])$, where the first two correspond to V_j^+ and V_{j-1}^+ respectively. The quotient function of $\tilde{\phi}$ maps $[f_1]$ to 1 and all other generators of one dimensional cones to zero. The quotient fan and function along the edge from v_j to v^- are similar.

Let us now check the vertex v_0 . Let us denote by $\bar{\ell}_j$, $\bar{\ell}^+$ and $\bar{\ell}^-$ the primitive generators of the one dimensional cones corresponding to the edges ℓ_j , ℓ^+ and ℓ^- respectively (remember that $\bar{\ell}^- = -\bar{\ell}^+$). Then $\tilde{\phi}$ has value -1 on each $\bar{\ell}_j$, zero on $\bar{\ell}^-$ and $-2n+1$ on $\bar{\ell}^+$. It is clear that the quotient fan along the directions $\bar{\ell}^+$ and $\bar{\ell}^-$ is just the normal fan of L and the quotient function of $\tilde{\phi}$ has value -1 on each generator of the one dimensional cones. Let us compute the quotient fan along $\bar{\ell}_j$. Observe that we have $\bar{\ell}_{j-1} = -\bar{\ell}_{j+1} - k_j\bar{\ell}_j$, since $\bar{\ell}_{j+1}$ is parallel to f_2 above, $\bar{\ell}_{j-1}$ is parallel to w and $\bar{\ell}_j$ points in the opposite direction with respect to f_1 . With this observation, we can compute that the quotient fan is given by the cones $\text{Cone}([\bar{\ell}_{j+1}], [\bar{\ell}^+])$, $\text{Cone}([\bar{\ell}_{j+1}], -[\bar{\ell}^+])$, $\text{Cone}(-[\bar{\ell}_{j+1}], -[\bar{\ell}^+])$ and $\text{Cone}(-[\bar{\ell}_{j+1}], [\bar{\ell}^+])$ and the quotient function maps $[\bar{\ell}_{j+1}]$ and $[\bar{\ell}^+]$ to zero, $-[\bar{\ell}^+]$ to $-2n+1$ and $-\bar{\ell}_{j+1}$ to $-k_j-2$. Thus the quotient functions of $\tilde{\phi}$ along the edge ℓ_j computed at the vertices v_j and v_0 coincide.

We now come to the vertex v^+ . Consider the normal fan of L in \mathbb{R}^2 and view it as embedded in \mathbb{R}^3 by the first two coordinates and denote $v = (0, 0, 1)$. Then the fan at v^+ is given by the cones $K_j^- = \text{Cone}(n\bar{\ell}_j - v, n\bar{\ell}_{j+1} - v, -v)$ and $K_j^+ = \text{Cone}(n\bar{\ell}_j - v, n\bar{\ell}_{j+1} - v, v)$, where $j = 1, \dots, k$. The cones K_j^- correspond to the polytopes V_j^+ , hence $-v$ points in the direction of v_0 and $n\bar{\ell}_j - v$ in the direction of v_j . Then $\tilde{\phi}$ has value n on $-v$ and 0 at all other generators of one dimensional cones. The quotient fan along the edge from v^+ to v_0 , i.e. along $-v$ can be seen to coincide with the normal fan of L and the quotient function maps $\bar{\ell}_j$ to -1 for all $j = 1, \dots, k$. Hence the quotient fan and functions along the edge from v^+ to v_0 coincide at the vertices v^+ and v_0 .

We now check the quotient along the edge from v^+ to v_j , i.e. along $n\bar{\ell}_j - v$. A basis for the quotient is $\{[\bar{\ell}_j], [\bar{\ell}_{j+1}]\}$. Notice that $[v] = n[\bar{\ell}_j]$, $[n\bar{\ell}_{j-1} - v] = -n(k_j + 1)[\bar{\ell}_j] - n[\bar{\ell}_{j+1}]$ and $[n\bar{\ell}_{j+1} - v] = n[\bar{\ell}_{j+1}] - n[\bar{\ell}_j]$. Therefore we have the four cones in the quotient: $\text{Cone}(-(k_j + 1)[\bar{\ell}_j] - [\bar{\ell}_{j+1}], -[\bar{\ell}_j])$, $\text{Cone}([\bar{\ell}_{j+1}] - [\bar{\ell}_j], -[\bar{\ell}_j])$, $\text{Cone}(-(k_j + 1)[\bar{\ell}_j] - [\bar{\ell}_{j+1}], [\bar{\ell}_j])$, $\text{Cone}([\bar{\ell}_{j+1}] - [\bar{\ell}_j], [\bar{\ell}_j])$, corresponding to the cones K_{j-1}^- , K_j^- , K_{j-1}^+ and K_j^+ respectively. Now the quotient function maps $-\bar{\ell}_j$ to 1 and all other generators of one dimensional cones to zero. This shows that the quotient fans and functions along the edge from v^+ to v_j coincide at the vertices v^+ and v_j .

The behavior of $\tilde{\phi}$ at the vertex v^- is completely analogous, as well as the computation of the quotient fans and functions. Also the quotient fan and function along the edge from v_j to v_{j+1} at the vertices v_j and v_{j+1} coincide, the computation is similar. Along all other edges, by the definition of $\tilde{\phi}$, the quotient function is trivially zero. Hence we have proved that $\tilde{\phi}$ is well defined.

Now we prove that the function $\check{\phi}'$ as defined in (39) is strictly convex. First let us recall some useful facts. Suppose Σ is a complete fan in \mathbb{R}^3 and ϕ a piecewise linear function on Σ . Let F be a two dimensional cone of Σ and C, C' two maximal cones such that $F = C \cap C'$. Then ϕ restricts to linear maps m and m' on C and C' respectively, such that $(m - m')|_F = 0$. Now consider a \mathbb{Z}^3 -basis $\{f_1, f_2, f_3\}$ of \mathbb{R}^3 , such that f_1 and f_2 generate the plane containing F and f_3 points towards the interior of C . Then the integer $h_\phi(F) = \langle m - m', f_3 \rangle$ is independent of the chosen basis. The function ϕ is strictly convex if and only if $h_\phi(F) > 0$ for every two dimensional cone F of Σ . Clearly $h_{N\phi}(F) = Nh_\phi(F)$ for every integer N and $h_{\phi+\phi'}(F) = h_\phi(F) + h_{\phi'}(F)$.

The fan at the vertex v_0 was unaffected by the subdivision $\check{\mathcal{P}}'$. Now recall that if we consider the space of not necessarily integral, piecewise linear functions on a fan Σ , then the strictly convex ones form an open cone inside this space. This implies that $\check{\phi}'$ is strictly convex at v_0 for sufficiently large N , since $\check{\phi}$ is strictly convex.

Now consider the vertices v_j . Here the fan structure has been changed by the subdivision. Let Σ be the fan before the subdivision and Σ' the new fan. A two dimensional cone F of Σ' can be of two types: it either intersects the interior of a maximal cone of Σ or it is entirely contained in a two dimensional cone of Σ . In the latter case we have $h_{\check{\phi}}(F) > 0$ and therefore for sufficiently large N ,

$h_{\tilde{\phi}}(F) = Nh_{\tilde{\phi}}(F) + h_{\tilde{\phi}}(F) > 0$. In the former case $h_{\tilde{\phi}}(F) = 0$, but it can be easily checked by direct calculation that $h_{\tilde{\phi}}(F) > 0$ for all such F 's. Therefore we have again $h_{\tilde{\phi}}(F) > 0$. The case for the vertices v^+ and v^- is analogous. This completes the proof. \square

Corollary 8.4. Suppose that the sets $K = \{p_1, \dots, p_k\}$ and $K' = \{p'_1, \dots, p'_r\}$ of positive nodes in B are each in a configuration like in Theorem 8.3 such that K and K' are the corners of S and S' respectively. Assume moreover that $p_1 = p'_1$ and that $S \cap S' = \{p_1\}$. Then the nodes $K \cup K'$ can be simultaneously resolved.

Proof. Let $P = L \times [0, m]$ and $P' = L' \times [0, m]$ be the polytopes of \mathcal{P} such that $S = L \times \{m/2\}$ and $S' = L' \times \{m/2\}$. Then P and P' have only one edge in common, i.e. the one containing p_1 . Denote it by e_1 . Notice that if we carry out the construction explained in Theorem 8.3 for P or for P' we obtain the same subdivision of the two-face \check{e}_1 , mirror of e_1 . Denote by v_1 and v_2 the opposite vertices of \check{e}_1 forming the diagonal in such a subdivision. Also denote by v_0 and w_0 the vertices mirror to P and P' respectively. Then the vertices of \check{e}_1 are v_1, v_2, v_0 and w_0 . Moreover, emanating from v_0 we have two edges ℓ^+ and ℓ^- and the integral points v^+ and v^- on them, as in the proof of the theorem. Let us denote by λ^+, λ^- and w^+ and w^- the analogous edges and points emanating from w_0 . Let Q_1^+ and Q_1^- be the polytopes such that $Q_1^+ \cap Q_1^- = \check{e}_1$. We subdivide Q_1^\pm by the polytopes $\text{Conv}(v_0, v_1, v_2, v^\pm)$, $\text{Conv}(w_0, v_1, v_2, w^\pm)$ and the closure of the complement these two. All other polytopes are subdivided as in Theorem 8.3 applied to P and P' . Moreover, the theorem applied to P and P' gives MPL-functions $\tilde{\phi}_1$ and $\tilde{\phi}_2$ on \check{B} respectively. Then we define

$$\check{\phi}' = N\check{\phi} + \tilde{\phi}_1 + \tilde{\phi}_2,$$

which, for sufficiently large N , is strictly convex. \square

We can also generalize these results to a configuration of nodes as depicted in Figure 18.

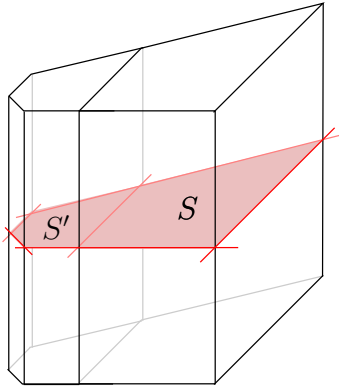


FIGURE 18.

Theorem 8.5. Suppose that the sets $K = \{p_1, \dots, p_k\}$ and $K' = \{p'_1, \dots, p'_r\}$ of positive nodes in B are each in a configuration like in Theorem 8.3 such that K and K' are the corners of S and S' respectively. Assume moreover that $S \cap S'$ is an edge of Δ having as vertices $p'_1 = p_1$ and $p_2 = p'_2$. Then the nodes $K \cup K'$ can be simultaneously resolved.

Proof. Let $P = L \times [0, m]$ and $P' = L' \times [0, m]$ be the polytopes of \mathcal{P} such that $S = L \times \{m/2\}$ and $S' = L' \times \{m/2\}$. Then, if q_1, \dots, q_k are the corners of L and q'_1, \dots, q'_r are the corners of L' we assume that the nodes p_j and p'_j correspond to $(q_j, m/2)$ and $(q'_j, m/2)$ respectively. Since we assume that $p_1 = p'_1$ and $p_2 = p'_2$, we have that $(q_1, m/2)$ and $(q_2, m/2)$ are identified with $(q'_1, m/2)$ and $(q'_2, m/2)$ via an identification of the corresponding 2 dimensional faces of P and P' respectively. Now observe that from the properties of positive nodes, it follows that we can assume that the two dimensional faces $L \times \{m\}$ and $L' \times \{m\}$ lie on the same affine 2-plane inside B , while the opposite faces $L \times \{0\}$ and $L' \times \{0\}$ lie on different planes (by the effect of monodromy around the common edge of S and S' , see Figure 8).

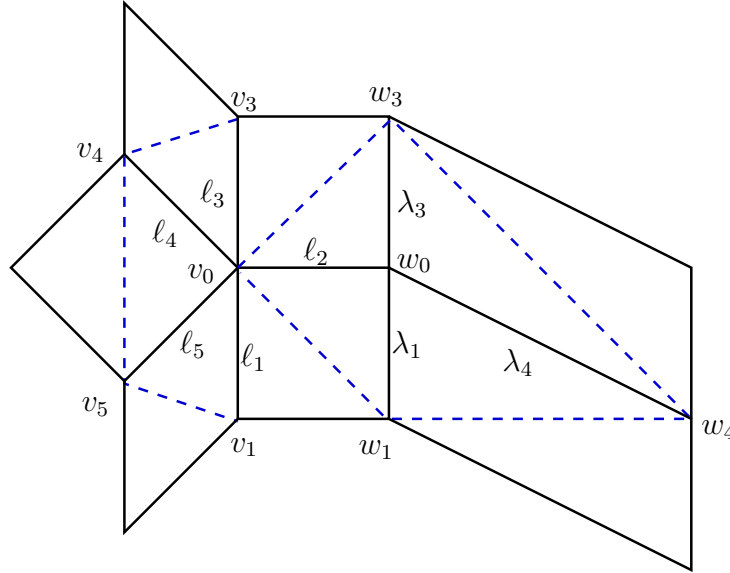


FIGURE 19.

Now let us look at the mirror. As in the proof of Theorem 8.3, we denote by Q_j^+ and Q_j^- the maximal polytopes of \mathcal{P}' corresponding to the vertices (q_j, m) and $(q_j, 0)$ respectively and by R_j^+ and R_j^- those corresponding to (q'_j, m) and $(q'_j, 0)$. Clearly we have $Q_j^\pm = R_j^\pm$ when $j = 1, 2$. Also denote by \check{e}_j and \check{d}_j respectively the square faces $Q_j^+ \cap Q_j^-$ and $R_j^+ \cap R_j^-$. We also denote the edges $\ell_j = \check{e}_{j-1} \cap \check{e}_j$ and $\lambda_j = \check{d}_{j-1} \cap \check{d}_j$. Clearly we have $\check{e}_j = \check{d}_j$ for $j = 1, 2$ and $\ell_2 = \lambda_2$. We let v_0 be the vertex dual to P and w_0 the vertex dual to P' . We have that v_0 and w_0 are connected by the edge ℓ_2 . The edges ℓ_j and λ_j

emanate from v_0 and w_0 respectively (see Figure 19) and we denote by v_j and w_j the other vertices of ℓ_j and λ_j respectively (here $j \neq 2$). We assume that v_0, w_0, w_3, v_3 are the vertices of \check{e}_2 and v_0, w_0, w_1, v_1 are the vertices of \check{e}_1 . We denote by ℓ^+ and ℓ^- (resp. λ^+ and λ^-) the other two edges emanating from v_0 (resp. w_0) which are mirror respectively to the faces $L \times \{m\}$ and $L \times \{0\}$ (resp. $L' \times \{m\}$ and $L' \times \{0\}$). We let v^+ and w^+ (resp. v^- and w^-) be the interior integral points on ℓ^+ and λ^+ (resp. ℓ^- and λ^-) which are closest to v_0 and w_0 respectively. Notice that we can assume that the edges ℓ^+ and λ^+ are parallel inside Q_2^+ (or Q_1^+). On the other hand, by effect of monodormy, ℓ^- and λ^- are not parallel inside Q_2^- .

We first construct a smoothing of the negative nodes lying inside the faces \check{e}_j and \check{d}_j . We do this in two steps. First subdivide the faces \check{e}_j and \check{d}_j , with $j \neq 1, 2$, by adding the diagonals from v_j to v_{j+1} and from w_j to w_{j+1} . We leave \check{e}_1 and \check{e}_2 as they are for the moment. We extend this subdivision as follows. The polytopes Q_j^\pm and R_j^\pm with $j \neq 1, 2$ are subdivided just like in the proof of Theorem 8.3, e.g. Q_j^+ is subdivided by taking $\text{Conv}(v_0, v_j, v_{j+1}, v^+)$ as one polytope and the closure of its complement as the other one. We subdivide Q_1^\pm (resp. Q_2^\pm) by taking $\text{Conv}(v_0, w_0, v^\pm, w^\pm, v_1, w_1)$ (resp. $\text{Conv}(v_0, w_0, v^\pm, w^\pm, v_3, w_3)$) as one polytope and the closure of its complement as the other. This is a well defined subdivision of $\tilde{\mathcal{P}}$, moreover we can assume that, perhaps after rescaling \tilde{B} , no parts of the discriminant $\tilde{\Delta}$ are affected by this subdivision other than the nodes we want to smooth.

We now define an MPL-function $\tilde{\phi}_1$. On the fan at v_0 (resp. w_0), $\tilde{\phi}_1$ takes the value -1 on the primitive, integral tangent vectors to the edges ℓ_j (resp. λ_j) and 0 on the primitive, integral tangent vector to ℓ_2 (resp. λ_2). On the primitive tangent vector to ℓ^+ (resp. λ^+), $\tilde{\phi}_1$ takes the value $-2n + 1$ and on the primitive tangent vector to ℓ^- (resp. λ^-) it takes the value 0 . On the fan at v_j (resp. w_j), $j \neq 2$, $\tilde{\phi}_1$ takes the value 1 on the primitive integral tangent vector to the edge ℓ_j (resp. λ_j) and zero on all other one dimensional cones. On the fan at v^\pm (resp. w^\pm), $\tilde{\phi}_1$ takes the value n on the primitive integral tangent vector to the edge ℓ^\pm (resp. λ^\pm) and zero on all other one dimensional cones. We let the reader check that $\tilde{\phi}_1$ is a well defined, MPL function with respect to the given decomposition.

The next step is to refine the above subdivision by further subdividing the polytopes Q_j^\pm with $j = 1, 2$. We denote by $\tilde{\mathcal{P}}'$ the resulting subdivision. For $k = 1$ or 3 , subdivide the polytopes $\text{Conv}(v_0, w_0, v_k, w_k, v^\pm, w^\pm)$ as

$$\text{Conv}(v_0, w_0, v_k, w_k, v^\pm, w^\pm) = \text{Conv}(v_0, w_0, w_k, w^\pm) \cup \text{Conv}(v_0, v_k, w_k, v^\pm, w^\pm).$$

This has the effect of subdividing the faces \check{e}_j , $j = 1, 2$, with the diagonal from v_0 to w_j (see Figure 19). Now notice that near w_0 the subdivision looks precisely as the one defined in the proof of Theorem 8.3, in case we wanted to resolve only the nodes p'_1, \dots, p'_r . So we can define a function $\tilde{\phi}_2$ to be zero outside the polytopes containing w_0 and just as in the proof of Theorem 8.3 on the polytopes which contain w_0 .

Finally, we define

$$\check{\phi}' = N\check{\phi} + \check{\phi}_1 + \check{\phi}_2,$$

for some positive integer N . As in Theorem 8.3, we can check that for sufficiently big N , $\check{\phi}'$ is strictly convex with respect to $\check{\mathcal{P}}'$. This completes the tropical smoothing of the given set of negative nodes. The discrete Legendre transform defines a tropical resolution of the mirror positive nodes. \square

We now prove that also some special configurations of negative nodes can be simultaneously resolved. Let (L, \mathcal{P}_L) be a two dimensional smooth tropical manifold with boundary, such that L is homeomorphic to a disc, and satisfying the following assumptions.

Assumption 8.6. Denote by v_1, \dots, v_k the boundary vertices of L and by $\lambda_1, \dots, \lambda_k$ the boundary edges, such that v_j and v_{j+1} are the vertices of λ_j . We assume the following:

- a) at every vertex v_j the affine structure is such that the edges λ_j and λ_{j-1} emanating from v_j are colinear.
- b) every vertex v_j belongs to only one other edge of L besides λ_j and λ_{j-1} . Denote it by γ_j .

Consider now $L \times [0, 1]$ with the product tropical structure and let $F_j = \lambda_j \times [0, 1]$ be the boundary 2-dimensional faces. Suppose there is a 3 dimensional tropical manifold (B, \mathcal{P}, ϕ) and an embedding of tropical manifolds $L \times [0, 1] \hookrightarrow B$ such that every two face F_j contains a negative node p_j of B . In particular $F_j \cap \Delta$ is the union of the two segments joining the barycenters of opposite pairs of edges of F_j . Clearly the negative nodes p_1, \dots, p_k are related by the tropical 2-cycle defined by $S = L \times \{\frac{1}{2}\}$. The vector field v in condition (c) of Definition 7.2 is parallel to edges of type $\{v_j\} \times [0, 1]$, where v_j is a vertex of L . Notice that the singular points of L generate edges of Δ which intersect S transversely in the interior, giving points as in conditions (b) and (i) of Definition 7.2. We have the following

Theorem 8.7. The negative nodes p_1, \dots, p_k in a configuration as above can be tropically resolved.

Proof. The proof is similar to Theorem 8.3. Since we want to use mirror symmetry to resolve the nodes we start by describing the mirror. Observe that, if $P \in \mathcal{P}_L$ is any j -dimensional polytope of L ($j = 0, 1, 2$), then the mirror of the $(j+1)$ -polytope $P \times [0, 1]$ of B will be a $(2-j)$ -polytope \check{P} , inside \check{B} . All the polytopes \check{P} are coplanar. We also have the polytopes $P \times \{1\}$ and $P \times \{0\}$ whose mirrors we denote by \check{P}^+ and \check{P}^- and their dimension is $3-j$. Clearly $\check{P}^+ \cap \check{P}^- = \check{P}$. To indicate polytopes of L we will use greek letters for edges ($\lambda, \delta, \gamma, \dots$), lower case for vertices (v, c, p, \dots) and upper case (C, F, N, \dots) for 2-faces. Every boundary edge λ_j of L belongs to a 2-face C_j of L . Moreover, it follows from point (b) of Assumption 8.6 that C_j intersects C_{j+1} in γ_{j+1} . Clearly the edge $\check{\gamma}_{j+1}$ joins the vertex \check{C}_j to the vertex \check{C}_{j+1} . All edges $\check{\gamma}_j$ enclose a region, homeomorphic to a disc, which we denote by \check{L} . Notice that if

P is a 2-face of L , then \check{P}^+ and \check{P}^- are edges emanating from \check{P} which lie on a common line passing through \check{P} and transversal to \check{L} . Moreover the edges of type \check{P}^+ (or \check{P}^-), with \check{P} a vertex of \check{L} , are pairwise parallel with respect to parallel transport inside \check{L} .

Given a boundary vertex v_j of L , \check{v}_j is a 2-face which contains $\check{\lambda}_{j-1}$, $\check{\gamma}_j$ and $\check{\lambda}_j$ as bounding edges. Moreover $\check{\lambda}_j = \check{v}_j \cap \check{v}_{j+1}$. Point (a) of Assumption 8.6 implies that $\check{\lambda}_{j-1}$ and $\check{\lambda}_j$ are parallel edges of \check{v}_j . Notice that $\check{\lambda}_j$ is mirror to the face F_j , therefore the barycenter of $\check{\lambda}_j$ is a positive node mirror to the node p_j . Moreover the line inside \check{v}_j going from the barycenter of $\check{\lambda}_{j-1}$ to the barycenter of \check{v}_j and then to the barycenter of $\check{\lambda}_j$ is part of the discriminant locus Δ .

In order to resolve the nodes p_1, \dots, p_k , we first define a new decomposition on $(\check{B}, \check{P}, \check{\phi})$ as follows. On each of the edges $\check{\lambda}_j$, \check{C}_j^+ and \check{C}_j^- emanating from \check{C}_j choose the integral points closest to \check{C}_j and call them t_j , q_j^+ and q_j^- respectively. Now subdivide \check{v}_j^+ in two convex polytopes, one being $W_j^+ = \text{Conv}(t_{j-1}, t_j, \check{C}_{j-1}, \check{C}_j, q_{j-1}^+, q_j^+)$ and the other one being the closure of its complement. Similarly subdivide \check{v}_j^- and let $W_j^- = \text{Conv}(t_{j-1}, t_j, \check{C}_{j-1}, \check{C}_j, q_{j-1}^-, q_j^-)$. Given any interior vertex v of L , we now give a subdivision of the 3-polytope \check{v}^+ . Notice that all vertices of \check{v} are of the type \check{C} , where C is 2-face of L containing v . Let q_C^+ (resp. q_C^-) be the integral point on \check{C}^+ (resp. \check{C}^-) closest to \check{C} . Consider the polytope inside \check{v}^+ given by the convex hull of all points \check{C} and q_C^+ for all 2-faces C containing v . Denote it by W_v^+ and subdivide \check{v}^+ by W_v^+ and the closure of its complement. Similarly subdivide \check{v}^- . This gives a well defined decomposition of \check{B} . Notice that W_v^+ , as a polytope, is just $\check{v} \times [0, 1]$.

Now we need to define a fan structure at the new vertices of the subdivision. This definition must have the effect of smoothing the positive nodes. Although the fan structure at \check{C}_j is unchanged, it is convenient to describe it. Let $\{e_1, e_2, e_3\}$ be an integral basis of $T_{\check{C}_j}\check{B}$ such that e_3 is tangent to \check{C}_j^+ , e_2 is tangent to $\check{\lambda}_j$ and e_1 is contained in the tangent wedge to \check{v}_{j+1} . There are pairs of integers (a_j, b_j) and (a_{j+1}, b_{j+1}) such that the tangent wedges to \check{v}_j^+ , \check{v}_{j+1}^+ , \check{v}_j^- , and \check{v}_{j+1}^- correspond respectively to

$$(40) \quad \begin{aligned} &\text{Cone}(-a_j e_1 + b_j e_2, e_2, e_3), \quad \text{Cone}(a_{j+1} e_1 + b_{j+1} e_2, e_2, e_3), \\ &\text{Cone}(-a_j e_1 + b_j e_2, e_2, -e_3), \quad \text{Cone}(a_{j+1} e_1 + b_{j+1} e_2, e_2, -e_3). \end{aligned}$$

Here $a_j e_1 + b_j e_2$ and $-a_{j+1} e_1 + b_{j+1} e_2$ are respectively the primitive generators of the cones corresponding to $\check{\gamma}_j$ and $\check{\gamma}_{j+1}$. Notice that a_j and a_{j+1} will be positive. We will not need the other cones of the fan structure at \check{C}_j .

Let us define the fan structure at t_j . There are eight maximal polytopes meeting at t_j : W_j^+ , W_{j+1}^+ , W_j^- , W_{j+1}^- and their complements. Let $\{f_1, f_2, f_3\}$ be the standard basis of \mathbb{R}^3 . We define the eight cones in the fan structure at t_j to be

$$\text{Cone}(a_{j+1} f_1 - b_{j+1} f_2, f_2, f_2 + f_3), \text{Cone}(-a_j f_1 - b_j f_2, f_2, f_2 + f_3),$$

$$\begin{aligned} & \text{Cone}(a_{j+1}f_1 - b_{j+1}f_2, f_2, -f_3), \text{Cone}(-a_jf_1 - b_jf_2, f_2, -f_3), \\ & \text{Cone}(a_{j+1}f_1 - b_{j+1}f_2, -f_2, f_2 + f_3), \text{Cone}(-a_jf_1 - b_jf_2, -f_2, f_2 + f_3), \\ & \text{Cone}(a_{j+1}f_1 - b_{j+1}f_2, -f_2, -f_3), \text{Cone}(-a_jf_1 - b_jf_2, -f_2, -f_3). \end{aligned}$$

where the first four correspond respectively to the tangent wedges of W_{j+1}^+ , W_j^+ , W_{j+1}^- and W_j^- . The other four to their complements. In the first cone, f_2 is tangent to the edge from t_j to \check{C}_j and therefore, by parallel transport inside W_{j+1}^+ , f_2 is a parallel to $-e_2$. The primitive integral tangent vector to the edge from t_j to q_j^+ corresponds, in the first cone, to $f_2 + f_3$. Notice that this implies that f_3 , by parallel transport inside W_{j+1}^+ , is parallel to e_3 . The vector $a_{j+1}f_1 - b_{j+1}f_2$ is tangent to the edge from t_j to t_{j+1} and therefore, by construction, it is parallel to $\check{\gamma}_{j+1}$ with respect to parallel transport inside W_{j+1}^+ . Similarly $-a_jf_1 - b_jf_2$, in the second cone, is tangent to the edge from t_j to t_{j-1} and it is therefore parallel to $\check{\gamma}_j$, with respect to parallel transport inside W_j^+ . Notice the crucial fact that in the third cone $-f_3$ is tangent to the edge from t_j to q_j^- and therefore, by parallel transport inside W_{j+1}^- , it is parallel to $-e_3 - e_2$. This choice of fan structure guarantees that parallel transport along a loop going from \check{C}_j to t_j passing inside W_{j+1}^+ and then back to \check{C}_j passing into W_j^+ is the identity. While moving along a loop going from \check{C}_j to t_j passing inside W_{j+1}^+ and then back to \check{C}_j passing into W_{j+1}^- gives the monodromy matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

computed with respect to the basis $\{e_1, e_2, e_3\}$. This corresponds to the smoothing of the positive node (compare with §6.4, in particular with the second construction of the smoothing).

We now define the fan structure at the points q_j^+ . The tangent wedges to W_j^+ , W_{j+1}^+ are mapped respectively to the cones:

$$\text{Cone}(-a_jf_1 + b_jf_2, f_2 + f_3, f_3), \text{Cone}(a_{j+1}f_1 + b_{j+1}f_2, f_2 + f_3, f_3).$$

Here f_3 is tangent to the edge from q_j^+ to \check{C}_j , $f_2 + f_3$ is tangent to the edge q_j^+ to t_j . In the first cone, $-a_jf_1 + b_jf_2$ is tangent to the edge from q_j^+ to q_{j-1}^+ . In the second cone $a_{j+1}f_1 + b_{j+1}f_2$ is tangent to the edge from q_j^+ to q_{j+1}^+ . The complements of W_j^+ and W_{j+1}^+ inside \check{v}_j^+ and \check{v}_{j+1}^+ are mapped respectively to,

$$\text{Cone}(-a_jf_1 + b_jf_2, f_2 + f_3, -f_3), \text{Cone}(a_{j+1}f_1 + b_{j+1}f_2, f_2 + f_3, -f_3).$$

The other polytopes meeting in q_j^+ come from the subdivision of \check{v}^+ , where \check{v} is a 2-face of \check{L} , containing the vertex \check{C}_j . Namely we have W_v^+ and its complement inside \check{v}^+ . Suppose that in the fan structure at \check{C}_j , the tangent wedge of \check{v} at \check{C}_j is $\text{Cone}(\alpha_1e_1 + \alpha_2e_2, \beta_1e_1 + \beta_2e_2)$. Then, at q_j^+ , the tangent wedges of W_v^+ and its complement are mapped respectively to the cones

$$\text{Cone}(\alpha_1f_1 + \alpha_2f_2, \beta_1f_1 + \beta_2f_2, f_3), \text{Cone}(\alpha_1f_1 + \alpha_2f_2, \beta_1f_1 + \beta_2f_2, -f_3).$$

This defines the fan structure at q_j^+ . Similarly we define the fan structure at q_j^- . The fan structure at points of type q_C^+ or q_C^- , where \tilde{C} is an interior vertex of \tilde{L} is defined similarly and we leave its definition to the reader. It can be verified that these fan structures are compatible with the fan structures at all other points which are unchanged. Moreover this construction produces a smoothing of the positive nodes mirror to p_1, \dots, p_k . In order to complete the resolution of p_1, \dots, p_k we need to find a suitable multivalued strictly convex piecewise linear $\tilde{\phi}'$. The strategy, as in the proof of Theorem 8.3, is to define a suitable $\tilde{\phi}$ which is “supported in a neighborhood of \tilde{L} ”. Then define $\tilde{\phi}'$ as in (39) and prove that it is strictly convex for large N . Here is how we define $\tilde{\phi}$. Let d be a positive integer. On the fan at t_j , let $\tilde{\phi}(f_2) = d$ and zero at all other primitive edges. On the fan at \tilde{C}_j let $\tilde{\phi}(e_2) = \tilde{\phi}(e_3) = -d$ and zero at all other primitive edges. On the fan at q_j^+ and q_j^- , let $\tilde{\phi}(f_3) = d$ and zero at all other primitive edges. We claim that for some choice of d , $\tilde{\phi}$ is a well defined multivalued piecewise linear function. Let us first explain how to find d . Notice that if we take $d = 1$, then $\tilde{\phi}$ may not be integral. For instance on the first cone of the list (40), we would get $\tilde{\phi}(e_1) = -b_j/a_j$ which may not be an integer. Similarly on the second cone $\tilde{\phi}(e_1) = b_{j+1}/a_{j+1}$. One can check that if we let d be a common multiple of a_j and a_{j+1} for all $j = 1, \dots, k$, then $\tilde{\phi}$ will be integral.

Verifying that $\tilde{\phi}$ is well defined is a tedious calculation similar to the one carried out in Theorem 8.3 and we therefore omit it. Also the argument to show that the function $\tilde{\phi}'$ defined in (39) is strictly convex for large N is the same. \square

9. EXAMPLES

We discuss some examples of tropical conifolds where various sets of nodes can be simultaneously resolved/smoothed giving rise to new smooth tropical manifolds. We know that some of these examples are associated to Batyrev and Borisov’s examples of Calabi-Yau threefolds as complete intersections in toric Fano manifolds. Some others probably are not, but we are not sure.

9.1. First example... As the set \mathcal{P} of polytopes we take 18 copies of a triangular prisms, i.e. of

$$(41) \quad T = \text{Conv}\{(0, 0, 0), (4, 0, 0), (0, 4, 0), (0, 0, 4), (4, 0, 4), (0, 4, 4)\}.$$

We divide \mathcal{P} in two families of nine prisms each and label prisms in each family by σ_{jk} and τ_{jk} respectively, where j, k are cyclic indices from 1 to 3. The vertices of $\sigma_{j-1, k-1}$ and $\tau_{j-1, k-1}$ are labeled like in Figure 20.

Notice that some polytopes will have vertices with the same label. We glue polytopes along 2-dimensional faces by matching vertices with the same label. For instance, Figure 21 shows the result of gluing τ_{11} , τ_{21} , τ_{31} . The result of identifying polytopes with this rule gives two (polyhedral) solid tori, one from each family. Now glue the boundaries of these solid tori along 2-dimensional faces by matching the vertex Q_{jk} with the vertex P_{jk} , for every $j, k = 1, 2, 3$.

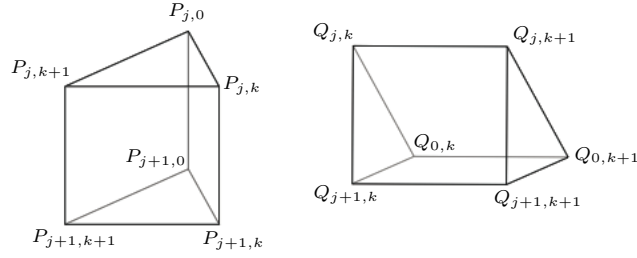


FIGURE 20. The two families of prisms: $\sigma_{j-1,k-1}$ on the left and $\tau_{j-1,k-1}$ on the right.

This gives the manifold B , homeomorphic to a 3-sphere, and the polyhedral decomposition \mathcal{P} . We now describe the fan structure at vertices. There are two

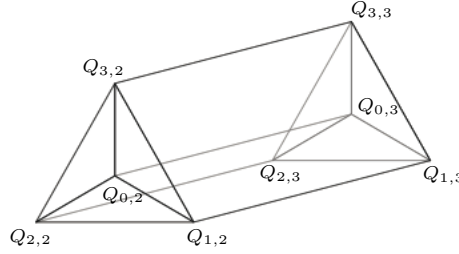


FIGURE 21. Assembling τ_{11} , τ_{21} , τ_{31} .

types of vertices, those labeled $P_{k,0}$ (or $Q_{0,k}$), lying in the interior of the solid tori, and those labeled $P_{j,k}$ (which are the same as $Q_{j,k}$) lying on the boundary of the solid tori. We take as representatives of each type, the vertices $P_{3,0}$ and $P_{3,1}$ respectively, and describe the fan structure there. Vertices of the same type will have the same fan structures, in the obvious sense.

For the vertex $P_{3,0}$, take the fan corresponding to $\mathbb{P}^2 \times \mathbb{P}^1$, whose three dimensional cones are

$$\text{Cone}(e_1, e_2, e_3), \text{Cone}(e_1, -e_1 - e_2, e_3), \text{Cone}(e_2, -e_1 - e_2, e_3),$$

$$\text{Cone}(e_1, e_2, -e_3), \text{Cone}(e_1, -e_1 - e_2, -e_3), \text{Cone}(e_2, -e_1 - e_2, -e_3).$$

The vertex $P_{3,0}$ belongs to six 3-dimensional faces of B . Three of these are $\sigma_{11}, \sigma_{12}, \sigma_{13}$, which intersect on the edge from $P_{3,0}$ to $P_{2,0}$. Identify the tangent wedges at $P_{3,0}$ of these three polytopes with the first three cones, in such a way that the tangent direction to the edge from $P_{3,0}$ to $P_{2,0}$ is mapped to e_3 . The other three 3-dimensional faces containing $P_{3,0}$ are $\sigma_{21}, \sigma_{22}, \sigma_{23}$, which intersect on the edge from $P_{3,0}$ to $P_{1,0}$. Identify the tangent wedges at $P_{3,0}$ of these three polytopes with the last three cones, in such a way that the tangent direction to the edge from $P_{3,0}$ to $P_{1,0}$ is mapped to $-e_3$.

For the vertex $P_{3,1}$, take the fan corresponding to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, whose 3-dimensional cones are the octants of \mathbb{R}^3 . Notice that there are eight 3-dimensional faces containing $P_{3,1}$: $\sigma_{23}, \sigma_{22}, \sigma_{12}, \sigma_{13}, \tau_{23}, \tau_{22}, \tau_{12}$ and τ_{13} .

The fan structure at $P_{3,1}$ identifies the tangent wedges of these eight polytopes respectively with: $\text{Cone}(-e_1, -e_2, e_3)$, $\text{Cone}(e_1, -e_2, e_3)$, $\text{Cone}(e_1, e_2, e_3)$, $\text{Cone}(-e_1, e_2, e_3)$, $\text{Cone}(-e_1, -e_2, -e_3)$, $\text{Cone}(e_1, -e_2, -e_3)$, $\text{Cone}(e_1, e_2, -e_3)$ and $\text{Cone}(-e_1, e_2, -e_3)$. This determines the fan structure in the neighborhood of $P_{3,1}$ and similarly for each vertex of the same type.

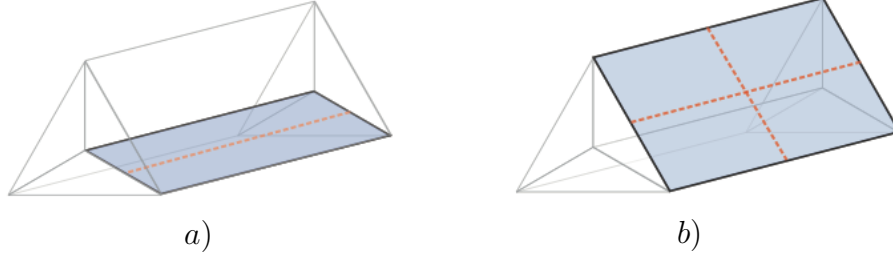


FIGURE 22. The discriminant locus Δ (dashed lines)

The discriminant locus Δ , also depicted in Figure 22, can be described as follows. Inside square faces with vertices $P_{j,0}, P_{j+1,0}, P_{j,k}$ and $P_{j+1,k}$ (or $Q_{0,k}, Q_{0,k+1}, Q_{j,k}$ and $Q_{j,k+1}$), Δ consists only of the segment joining the barycenter of the edge from $P_{j,0}$ to $P_{j,k}$ (resp. from $Q_{0,k}$ to $Q_{j,k}$) to the barycenter of the edge from $P_{j+1,0}$ to $P_{j+1,k}$ (resp. from $Q_{0,k+1}$ to $Q_{j,k+1}$), see Figure 22 (a). Monodromy around this component of Δ is given by the matrix

$$(42) \quad \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Observe that all these segments together give six disjoint circles, three in the interior of each solid torus. Moreover each circle has multiplicity 3, as can be seen from the monodromy. It can be shown that the structure can be modified slightly so that each circle splits into three, each one with the monodromy of generic-singularities. We will ignore this issue here.

Inside square faces with vertices $P_{j,k}, P_{j,k+1}, P_{j+1,k+1}$ and $P_{j+1,k}$, Δ has a quadrivalent vertex (see Figure 22 (b)). In fact it can be checked that this vertex is a negative node. Overall there are 9 negative nodes. There are no other components of Δ .

Finally we define a strictly convex piecewise linear function ϕ on (B, \mathcal{P}) . It is enough to specify the function on the fans associated to each vertex. At vertices of type $P_{k,0}$ (resp. $Q_{0,k}$) the fan is the fan of $\mathbb{P}^2 \times \mathbb{P}^1$. We only need to specify the values of ϕ at primitive generators of one dimensional cones. So we define ϕ to have value 1 at $-e_1 - e_2$ and at e_3 , and zero at the remaining ones. At vertices of type $P_{j,k} = Q_{j,k}$, the fan is the fan of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Here we take ϕ to have value 1 at $-e_1, -e_2$ and $-e_3$ and zero at the remaining ones. One can check that (B, \mathcal{P}, ϕ) is a well defined tropical conifold with 9 negative nodes.

9.2. ...and its mirror. The discrete Legendre transform of the previous example gives its mirror $(\check{B}, \check{\mathcal{P}})$. The polytopes dual to the vertices $P_{j,0}$ or $Q_{0,k}$ are six triangular prisms (i.e. $\text{Conv}\{(0, 0, 0), (0, 1, 0), (1, 0, 0), (0, 0, 1), (0, 1, 1), (1, 0, 1)\}$), which we label σ_j and τ_k respectively. The polytopes dual to the other 9 vertices are cubes (i.e. $\text{Conv}\{(0, 0, 0), (0, 1, 0), (1, 0, 0), (1, 1, 0), (0, 0, 1), (1, 0, 1), (0, 1, 1), (1, 1, 1)\}$). Let us denote the nine cubes by ω_{jk} . We label the vertices of these polytopes by the letters E_{jk} and F_{jk} as in Figure 23. The 2-dimensional faces of these

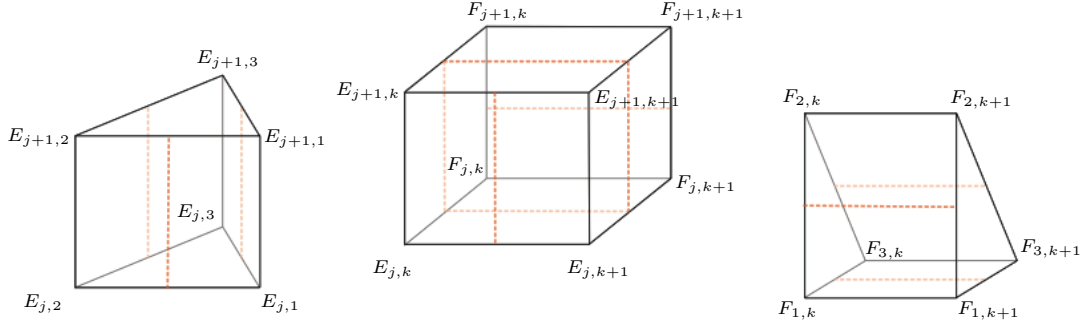


FIGURE 23. The polytopes σ_j (left), ω_{jk} (center) and τ_k (right). The dashed red lines form the discriminant locus Δ

polytopes are glued by matching the vertices with the same labeling. Notice that by first gluing together σ_1, σ_2 and σ_3 on the one hand and τ_1, τ_2 and τ_3 on the other, we obtain a pair of solid tori. Then, in Figure 23, the front faces of the cubes ω_{jk} are glued to the square faces of σ_j and the back faces are glued to the square faces of τ_k . The final result is again a 3-sphere.

We now need to specify the fan structure at vertices. The fan at every vertex is the fan of $\mathbb{P}^2 \times \mathbb{P}^1$. In the fan structure at $E_{j,k}$ the primitive tangent vectors to the five edges going to $E_{j,k+1}$, to $E_{j,k-1}$, to $F_{j,k}$, to $E_{j+1,k}$ and to $E_{j-1,k}$ are mapped respectively to $e_1, e_2, -e_1 - e_2, e_3$ and $-e_3$. Similarly (and symmetrically) we have the fan structure at the vertex $F_{k,j}$. By inspection one can see that the discriminant locus Δ is as depicted in Figure 23. In fact monodromy around the edges of Δ contained in the union of the σ_j 's (or in the union of the τ_j 's) is conjugate to (42). The remaining edges of Δ , those which do not intersect the triangular prisms, have standard generic-singular monodromy. Notice that each edge going from $E_{j,k}$ to $F_{j,k}$ contains a positive node. So that $(\check{B}, \check{\mathcal{P}})$ has 9 positive nodes, which are mirror to the 9 negative ones in (B, \mathcal{P}) . We also have a strictly convex piecewise linear function $\check{\phi}$.

9.3. Resolving/smoothing nodes. We now apply the results of Section 8 to simultaneously resolve and smooth certain sets of nodes in these two examples. In \check{B} , let us denote by $\ell_{j,k}$ the edge from $E_{j,k}$ to $F_{j,k}$, which contains a positive node. Notice that we can cut each cube with the unique plane passing through the four nodes, this gives us a tropical 2-cycle S containing the nodes. Thus every quadruple of nodes contained in a cube is related in the sense of Definition 8.1, moreover they are also in a configuration where we can apply Theorem 8.3,

thus they can be resolved. In fact any subset of the 9 nodes which is a union of quadruples of nodes contained in a cube is in a configuration which can be resolved with the criteria of Section 8. For instance if we take the six nodes inside two cubes sharing a common face (e.g. $\omega_{j,k}$ and $\omega_{j,k+1}$), then these are in a configuration like in Theorem 8.5. Similarly the seven nodes inside a pair of cubes sharing an edge can be resolved using Corollary 8.4. We can also resolve all 9 nodes, by observing that we can find tropical 2-cycles S_1 , S_2 and S_3 which are the squares obtained by cutting the cubes ω_{11} , ω_{22} and ω_{33} . The pairwise intersection of the S_j 's is just a point, and therefore Corollary 8.4 applies. The last case is given by 8 nodes. These can also be resolved with a slight refinement of the arguments above.

Let us now discuss the smoothing of the nodes in \tilde{B} . By mirror symmetry, smoothing nodes corresponds to resolving the mirror ones. So let us look at (B, \mathcal{P}) . Consider for instance the polytopes $\tau_{11}, \tau_{21}, \tau_{31}$ as in Figure 21. Then the three square faces of these polytopes which do not contain the vertices $Q_{0,2}$ and $Q_{0,3}$ have a negative node in their barycenter (see also Figure 22 (b)).

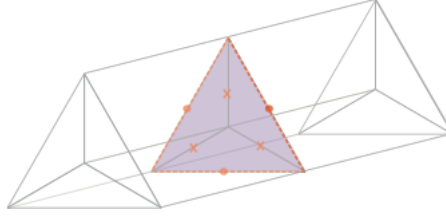


FIGURE 24. The tropical 2-cycle which relates 3 nodes in B .

We now construct a tropical 2-cycle containing these three nodes (see Figure 24). Inside the triangular prism T , given by (41), consider the triangle $T \cap \{z = 2\}$, obtained by cutting T in the middle by a plane parallel to the triangular faces. Then define the tropical 2-cycle S as the union of the copies of this triangle contained in τ_{1k}, τ_{2k} and τ_{3k} . As the vector field v of Definition 7.2 we take the parallel transport along S of the tangent vector to the edge from $Q_{0,2}$ to $Q_{0,3}$. Notice that $S \cap \Delta$ consists of the boundary of S and three other points in the interior. It can be easily verified that the latter points satisfy condition (i) of Definition 7.2. Also all other conditions of Definition 7.2 are satisfied. It can be also verified that Theorem 8.7 applies to this configuration. The same thing can of course be said about the three nodes in the union of σ_{j1}, σ_{j2} and σ_{j3} . Thus the sets of three nodes in these configurations can be resolved and the corresponding mirror nodes in \tilde{B} can be smoothed. Notice that the mirror nodes are those which are in the barycenters of the edges $\ell_{j,1}, \ell_{j,2}$ or $\ell_{j,3}$ (or $\ell_{1,k}, \ell_{2,k}$ or $\ell_{3,k}$).

Inside B , let us consider the nine square faces containing the nine nodes. These form the boundary of the solid torus formed by the polytopes τ_{jk} (or σ_{jk}). Represent the boundary as a big square subdivided in nine small ones. Then Figure 25 represents the various possibilities we have of resolving and smoothing

nodes in order to obtain a smooth tropical manifold. The families of nodes which can be resolved are those which lie on the horizontal or vertical lines of the grid and these are represented by nodes which are circled (in blue). The nodes which are being smoothed are the ones whose faces are subdivided by (blue) diagonal lines. In fact these lines represent the subdivision given by Theorem 8.5. As we can see we have 4 cases: we can resolve all nine nodes; smooth 4 and resolve 5; smooth 6 and resolve 3 or smooth all 9 nodes. Thus we have four different, smooth tropical manifolds. The Gross-Siebert reconstruction theorem gives four different Calabi-Yau manifolds. Let us denote them respectively by X_0 , X_4 , X_6 and X_9 . Now let us consider the mirror manifolds \check{X}_0 , \check{X}_4 , \check{X}_6 and \check{X}_9 . These are obtained from \check{B} respectively by smoothing all nine nodes; resolving 4 and smoothing 5; resolving 6 and smoothing 3 or resolving all 9 nodes. We know that \check{X}_0 corresponds to an example of Schoen's Calabi-Yau [27]. This can be described as follows. Let $f_1 : Y_1 \rightarrow \mathbb{P}^1$ and $f_2 : Y_2 \rightarrow \mathbb{P}^1$ be two rational elliptic surfaces with a section such that for no point $x \in \mathbb{P}^1$, $f_1^{-1}(x)$ and $f_2^{-1}(x)$ are both singular. Then Schoen's Calabi-Yau is the fibred product $Y_1 \times_{\mathbb{P}^1} Y_2$. It was proved in [18] that this Calabi-Yau can be represented as a complete intersection inside $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ of hypersurfaces of tridegree $(1, 3, 0)$ and $(1, 0, 3)$. Later Gross showed in [9] that the associated tropical manifold is precisely \check{B} with all nine nodes smoothed (see also [16] for similar methods). So we expect that \check{X}_0 is homeomorphic to Schoen's Calabi-Yau (see Theorem 0.1 in [9]). Thus we have $b_2(\check{X}_0) = 19$, $b_3(\check{X}_0) = 40$ and $\chi(\check{X}_0) = 0$. Now \check{X}_4 , \check{X}_6 and \check{X}_9 are related to \check{X}_0 by a conifold transition at respectively 4, 6 and 9 Lagrangian spheres in \check{X}_0 . Therefore, applying (19), we obtain $\chi(\check{X}_4) = 8$, $\chi(\check{X}_6) = 12$ and $\chi(\check{X}_9) = 18$. Applying the method described in [9], we have

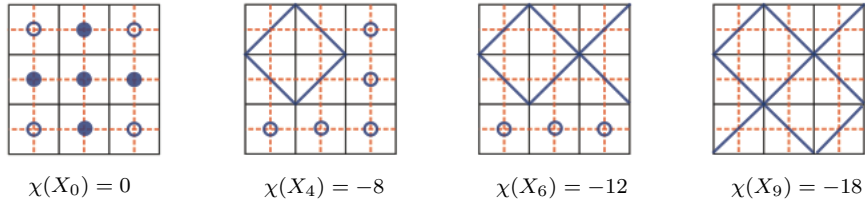


FIGURE 25. Smoothings and resolutions: the blue lines are the subdivisions required by the smoothing. The blue circles indicate the nodes being resolved.

computed that the tropical manifold obtained from B by smoothing all nodes corresponds to a complete intersection in $\mathbb{P}^3 \times \mathbb{P}^3$ of polynomials of bidegree $(3, 0)$, $(0, 3)$ and $(1, 1)$. Therefore we expect X_9 to be homeomorphic to such a manifold. We have $b_2(X_9) = 14$, $b_3(X_9) = 48$ (see also [21]). Obviously its mirror \check{X}_9 has $b_2(\check{X}_9) = 23$ and $b_3(\check{X}_9) = 30$. Since \check{X}_9 is related to \check{X}_0 by a conifold transition at 9 Lagrangian spheres in \check{X}_0 , from (19) we obtain $c = 5$ and $d = 4$. Therefore the vanishing cycles span a space of dimension 5 in $H_3(\check{X}_0)$. We believe, although we have not proved it, that five linearly independent Lagrangian spheres correspond to the nodes (in the mirror \check{B}) which are marked

by a thick circle in Figure 25. Indeed, in the previous discussion, we observed that four nodes contained in a square (of the dashed grid in Figure 25) are related by one relation. Therefore we expect that their vanishing cycles span a space of dimension 3. Observe that all 9 vanishing cycles can be obtained from the given 5 using these relations. Using these facts and (19), we conjecture that

$$\begin{aligned} b_2(\check{X}_4) &= b_2(\check{X}_0) + 1 = 20, & b_3(\check{X}_4) &= b_3(\check{X}_0) - 6 = 34 \\ b_2(\check{X}_6) &= 21, & b_3(\check{X}_6) &= 32 \end{aligned}$$

Moreover we also have the mirrors X_0 , X_4 , X_6 and X_9 .

It is likely that \check{X}_4 , \check{X}_6 and \check{X}_9 can be obtained as conifold transitions by classical methods from the equations defining \check{X}_0 . However we do not know if the corresponding tropical manifolds can be obtained from known toric degenerations, such as in toric Fano manifolds. Moreover we do not know if the mirrors have ever been computed by more standard methods.

9.4. More examples. Here we generalize the above example. For every pair of integers (L, M) , ranging from 3 to 6, we construct a tropical conifold as follows. Take $2LM$ copies of the triangular prism T considered in (41) and divide them in two families each containing LM copies. We denote the two families by σ_{jk} and τ_{jk} where j, k are cyclic indices of order L and M respectively. Now, to form B we do the same as above: we label the vertices of these prisms like in Figure 20 and we glue the 2-dimensional faces by matching the vertices with the same labels. Clearly, for $L = M = 3$ we obtain the same as above. Observe that assembling $\tau_{11}, \tau_{21}, \dots, \tau_{L1}$ with the above rule looks like the pictures represented in Figure 26 multiplied by the interval $[0, 4]$. Similarly we can say about $\sigma_{11}, \sigma_{12}, \dots, \sigma_{1M}$, with L in place of M . As above, the union of all the σ_{jk} 's on the one hand and of all the τ_{jk} 's on the other gives two solid tori which are again glued together to form a 3-sphere.

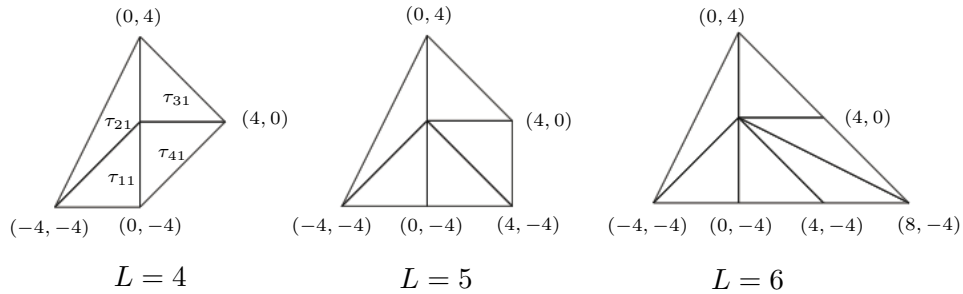


FIGURE 26.

The fan structures at vertices are defined in a very similar way to the example above. In fact at points of type $P_{j,k}$, $k \neq 0$, the fan structure is exactly the same as in the example above, i.e. it is the fan of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. At a point of type $Q_{0,k}$ (or $P_{j,0}$) it is also like in the example above if $L = 3$ (resp. if $M = 3$). Otherwise we define it as follows. For integers $L = 4, 5, 6$, consider the fan Σ_L in \mathbb{R}^2 which can be seen at the common point (i.e. the origin) of all simplices

in Figure 26. Clearly Σ_L has L 2-dimensional cones. Denote them in clockwise order by C_1, \dots, C_L , where C_1 is the one generated by $\{(-1, -1), (0, 1)\}$. Then form the fan in \mathbb{R}^3 whose cones are $C_j^+ = C_j \times [0, +\infty)$ and $C_j^- = C_j \times (-\infty, 0]$. Now, the 3-dimensional polytopes which contain the point $Q_{0,k}$ are $\tau_{j,k-1}$ and $\tau_{j,k-2}$, with $j = 1, \dots, L$. The fan structure at $Q_{0,k}$ identifies the tangent wedge of $\tau_{j,k-1}$ with C_j^+ and of $\tau_{j,k-2}$ with C_j^- . Similarly we can define the fan structure at points of type $P_{j,0}$ but with M in place of L and the roles of j and k inverted. Then we can also define a strictly convex piecewise linear function ϕ , just by suitably choosing one on each of the two types of fans. This defines our tropical conifold (B, \mathcal{P}, ϕ) , depending on the choice of integers $L, M = 3, 4, 5, 6$. Notice that we cannot go beyond 6 in this construction, because the polytopes in Figure 26 would lose convexity at the point $(4, 0)$ and the tropical manifold would not be smooth in the sense of §2.9. Discrete Legendre transform gives the mirror $(\check{B}, \check{\mathcal{P}}, \check{\phi})$. Notice that again B has a negative node on every square face that does not contain a point of type $P_{j,0}$ or $Q_{0,k}$, i.e. square faces that are on the boundary of the solid tori. Therefore, there are LM negative nodes.

Notice that in $(\check{B}, \check{\mathcal{P}}, \check{\phi})$, the polytope mirror to a point $P_{j,k}$, $k \neq 0$, is a cube. Just as in the above example, this cube contains 4 positive nodes which are related in the sense of Definition 8.1 and can be simultaneously resolved using Theorem 8.3. The negative nodes in B which are mirror to these 4 nodes are those contained in the prisms $\sigma_{j-1,k-1}, \sigma_{j-1,k-2}, \sigma_{j-2,k-1}, \sigma_{j-2,k-2}$. These can be simultaneously smoothed. Notice that for every fixed j (or k) the M (resp. L) nodes contained in the prisms $\sigma_{j,1}, \dots, \sigma_{j,M}$ (resp. $\sigma_{1,k}, \dots, \sigma_{L,k}$) are also related (see the example above), therefore they can be simultaneously resolved. Depending on the choices of nodes to be simultaneously smoothed/resolved we get diagrams similar to those in Figure 25, but on an $L \times M$ grid. In Figure 27 we have represented the diagram for three new examples in the cases (L, M) is the pair $(3, 4), (3, 5)$ or $(3, 6)$. In B these correspond to smoothing 8, 10 or 12 nodes respectively and resolving the others. We believe that by smoothing all nodes in \check{B} we still get Schoen's Calabi-Yau. Therefore, with a similar argument as above, we conjecture that $b_2(X_8) = 14$, $b_3(X_8) = 46$, $b_2(X_{10}) = 13$, $b_3(X_{10}) = 48$, $b_2(X_{12}) = 12$, $b_3(X_{12}) = 50$.

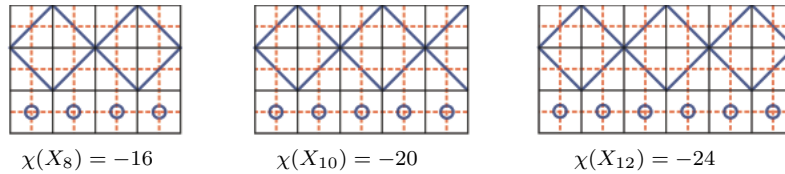


FIGURE 27.

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